## DOCUMENT RESUME

ED 173 148 SE 028 243

| | |
|---|---|
| AUTHOR | Allen, Frank B.; And Others |
| TITLE | Mathematics for High School, Intermediate Mathematics (Part 2). Commentary for Teachers. Preliminary Edition. |
| INSTITUTION | Stanford Univ., Calif. School Mathematics Study Group. |
| SPONS AGENCY | National Science Foundation, Washington, D.C. |
| PUB DATE | 59 |
| NOTE | 193p.; For related documents, see SE 028 244 and ED 135 627-628; Contains occasional light and broken type |
| EDRS PRICE | MF01/PC08 Plus Postage. |
| DESCRIPTORS | *Algebra; Curriculum; *Curriculum Guides; *Instruction; Mathematics Education; *Number Systems; Secondary Education; *Secondary School Mathematics |
| IDENTIFIERS | *Complex Numbers; *School Mathematics Study Group |

ABSTRACT

This is part two of a three-part manual for teachers using SMSG high school text materials. Each chapter contains a commentary on the text, answers to exercises, and suggested test questions. Chapter topics include quadratic equations, the complex number system, equations of the second degree, and systems of first degree equations. (MP)

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# MATHEMATICS FOR HIGH SCHOOL

## INTERMEDIATE MATHEMATICS (Part 2)

## *Commentary for Teachers*

(preliminary edition)

2

SE 028 243

# MATHEMATICS FOR HIGH SCHOOL

## INTERMEDIATE MATHEMATICS (Part 2)

## *Commentary for Teachers*

(preliminary edition)

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*Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.*

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ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

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# Teachers Manual

## Chapter 4

## Quadratic Equations

### Introduction

The primary objective of this chapter is to develop further skill in solving quadratic equations. The range of difficulty goes from very simple factorable equations to those which need to be changed into recognizable quadratic form.

The first part of the chapter should serve as a review for the more able students; however, adequate time should be given on this part to those less able.

Graphing is used to supplement the algebraic methods employed to find the solution set of quadratic equations and quadratic inequalities. [Note, that the solution of inequalities in Chapter I is now extended to the solution of quadratic inequalities in one and two variables]. Graphing is used to enrich the student's understanding of the nature of the roots as portrayed by the discriminant.

In the process of seeking roots, emphasis is given to the domain of definition, especially for fractional and radical equations.

The only starred section is The Remainder Theorem. This section may be used with the more able classes or with bright individual students for enrichment purposes. Another source for supplementary material will be found in this manual. Some of these topics are the same as found in this textbook, but with a more mature treatment; while, there are others that are not found in traditional textbooks.

## 4 - 1 Comments

This section gives the connecting link between the zeros of the function, the roots of the equation, and the intercepts of the graph.

An interesting means for constructing the table is given in the topic, Synthetic Substitution at the end of the manual. The portion of the graph that is of particular interest lies between the upper and lower bounds, discussed in a separate note.

When the textbook mentions the solution of equations, you may wish to introduce some set language. For instance, in speaking of the roots of $x^2 - 5x - 6 = 0$, you may wish to call to the attention of the students that the solution set $\{6,-1\}$ consists of 6 and - 1, Another way one may describe the solution set is,

$$\{x : x = 6 \text{ or } x = -1\}, \quad \text{or} \quad \{x | x = 6 \text{ or } x = -1\}$$

read "The set of all numbers x such that x is 6 or x is -1."

On the other hand, the solution set of $y = x^2 - 5x - 6$ may be described as,

$$\{(x,y) : y = x^2 - 5x - 6\}.$$

This is read, "the set of ordered pairs (x,y), such that $y = x^2 - 5x - 6$."

Point out to the student, that the graph of the solution set, $\{x : x^2 - 5x - 6 = 0\}$ is on the number line; while that of the solution set, $\{(x,y) : y = x^2 - 5x - 6\}$ is on the number plane. Suggest that your students compare these two graphs.

Exercises 4 - 1. Answers

1) We will obtain the graph of equation $y = x^2 + 3x - 10$ from a table of values

| x | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 8 | 0 | -6 | -10 | -12 | -12 | -10 | -6 | 0 | 8 |

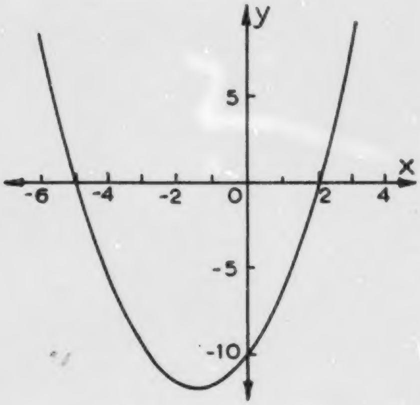


Figure 4-1b

As may be seen from the graph, zeros of the function f, defined by $y = x^2 + 3x - 10$, are -5 and 2. These numbers are also the roots of the quadratic equation

$$x^2 + 3x - 10 = 0$$

2) Graph the function $y = -x^2 + 4$ from a table of values.

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| y | -5 | 0 | 3 | 4 | 3 | 0 | -5 |

The zeros of the function are -2 and 2. Note that these are the roots of the quadratic equation $-x^2 + 4 = 0$.

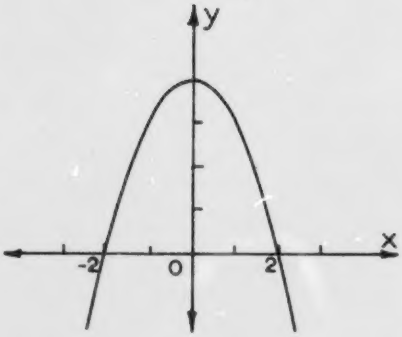
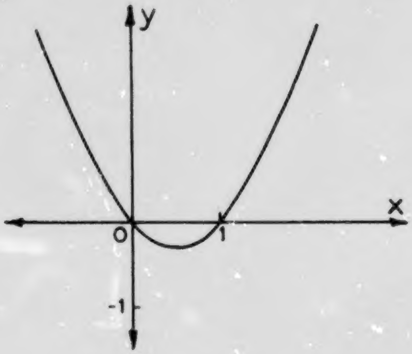


Figure 4-1c

3)



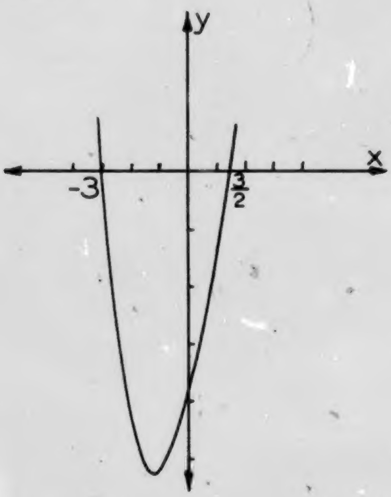
Draw the graph of $y = x^2 - x$ by, (a) constructing a table as done in 1) and 2), (b) completing the square to get the equation in the form $y = (x-k)^2 + p$ so as to determine the axis of symmetry and the turning point

$$y = (x^2 - x + \frac{1}{4}) - \frac{1}{4}$$

$$y = (x - \frac{1}{2})^2 - \frac{1}{4}$$

Intercepts: 0 and 1

4)



$$y = 3x^2 + 7x - 6$$

$$y = 3(x^2 + \frac{7}{3}x + \frac{49}{36}) - 6 - 3(\frac{49}{36})$$

$$y = 3(x + \frac{7}{6})^2 - \frac{127}{12}$$

Vertex at $-\frac{7}{6}, -10\frac{7}{12}$

Let $y = 0$

$$3x^2 + 7x - 6 = 0$$

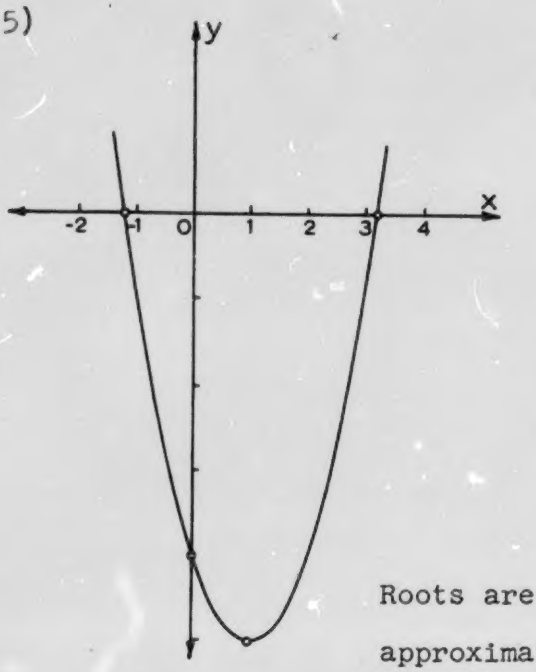
$$(3x - 2)(x + 3) = 0$$

$$x = -3$$

$$x = \frac{3}{2}$$

Zeros are $\frac{3}{2}$ and -3.

5)



Roots are irrational,

approximately -1.2 and 3.2

6)

| x | y |
|---|---|
| -5 | 10 |
| -4 | -1 |
| -3 | -8 |
| -2 | -11 |
| -1 | -10 |
| 0 | -5 |
| 1 | 4 |

Insist on the pupils drawing the graph. It will show that one intercept lies between -5 and -4, nearer -4; the other lies between -1 and 0.

4 - 2. Comments

In Chapter 1, Exercises 1 - 7 of Exercise 4, the equivalence of the transformations which we use here was proved. This Chapter gives them in words, whereas Chapter 1 gives them in terms of variables. You may wish to state them in the variable form.

*Transformation of Equations*. [In this additional note we examine the significance of adding and multiplying equations using the notion of their solution sets.]

Let $A$ be the set of roots for one equation $f_1(x) = 0$, and let $B$ be the set of roots for another, $f_2(x) = 0$, Then the *set union* $A \cup B$ will be the solution set for the "composite" equation $f_1(x) \; f_2(x) = 0$. Multiplying two equations *whose right members are zero* has the effect of combining their roots. On the other hand if we solve $f_1(x) = 0$ and $f_2(x) = 0$ simultaneously, seeking values for $x$ which satisfy both, the common roots belong to the *intersection set* $A \cap B$. The roots of $f_1 f_2 = 0$ are roots of $f_1 = 0$ or $f_2 = 0$, the roots of the system $f_1 = 0$, $f_2 = 0$ are roots of $f_1 = 0$ *and* $f_2 = 0$. If we *add* two equations the roots of $f_1 + f_2 = 0$ must include the common roots.

Since the solution set of $f_1 = 0$ is a subset of the solution set of $uf_1 = 0$, and the solution set of $f_2 = 0$ is a subset of the solution set of $vf_2 = 0$, therefore, the common solution set of the system $f_1 = 0$, $f_2 = 0$ are among the solutions of $uf_1 + vf_2 = 0$.

We consider a couple of examples showing the power of these set notions. Let $f_1 = 0$ be the equation $y^2 - x - 6y + 9 = 0$ whose solution set are the points $(x, y)$ on its graph; and let $f_2 = 0$ be the equation $x^2 - 10x - 8y + 25 = 0$. The graphs are both parabolas and they intersect in 4 points $A, B, C, D$. It is required to prove that $A, B, C, D$ lie on the same circle. If we eliminate $y$ we get a quartic equation for $x$, namely

$(x - 1)(x^3 - 19x^2 + 83x - 1) = 0$. Although the rational root gives the point (1,2) the roots of the cubic are irrational.

Now observe how our set theorem that "the common solutions of $f_1 = 0$, $f_2 = 0$ are among the solutions of $uf_1 + vf_2 = 0$ disposes of the problem. We add the two equations (choosing $u = v = 1$).

$$\begin{array}{r} y^2 - x - 6y + 9 = 0 \\ \underline{x^2 - 10x - 8y + 25 = 0} \\ x^2 + y^2 - 11x - 14y + 34 = 0 \end{array}$$

This is the equation of a circle, as can readily be seen by completing the square,

$$(x - \tfrac{11}{2})^2 + (y - 7)^2 = (\tfrac{11}{2}) + (7)^2 - 34$$

and taking the square root; it is the locus of points (x,y) whose distance from $(\frac{11}{2}, 7)$ is a constant. Not only have we proved that A,B,C,D lie on the same circle, but we have found the equation of that circle!

As our second example we seek the circle which circumscribes the triangle ABC whose sides satisfy three linear equations, say

BC: $x + 3y - 5 = 0$, AC: $2x + y - 10 = 0$, AB: $x - 7y + 25 = 0$.

When the equation

$$\frac{k_1}{x + 3y - 5} + \frac{k_2}{2x + y - 10} + \frac{k_3}{x - 7y + 25} = 0.$$

is cleared of fractions it will represent a curve passing through A,B, and C no matter how $k_1$, $k_2$, and $k_3$ are chosen. If they are chosen to eliminate the xy-term and to make the coefficients of $x^2$ and $y^2$ equal, the graph will be a circle. [The points are A(3,4), B(-4,3), C(5,0) and the circle is $x^2 + y^2 = 25$].

Oral Exercises - Answers

1. $f(x) = x^2 + 8x + 15 = 0$
2. $f(x) = x^2 + x - 6 = 0$
3. $f(x) = x^2 - 14x + 49 = 0$
4. $f(x) = x^2 + 1 = 0$
5. $f(x) = 4x^2 + 8x - 5 = 0$

Exercises 4 - 2 Answers

The following exercises contain pairs of equivalent equations: 1,2,3,8,9.

One may question the inclusion of 9 in the set indicating equivalent equations. Observe that the two equations here have only one real root for each, and this is 0. Therefore, by definition, in the domain of real numbers, they are equivalent.

## 4 - 3. Comments

Solving quadratic equations by the factoring method uses the theorem:

$$A \cdot B = 0 \text{ if, and only if, } A = 0 \text{ or } B = 0.$$

In giving the solution of

$$x^2 - 3x + 2 = 0,$$

we write $(x - 2)(x - 1) = 0$ to get $x^2 - 3x + 2 = 0$ in the form $A \cdot B = 0$.

Now we write,

$$x - 2 = 0 \quad \text{or} \quad x - 1 = 0, \quad \text{which says} \quad A = 0 \text{ or } B = 0$$

$$x = 2 \quad \text{or} \quad x = 1$$

But the roots are 2 and 1. Other ways to give the solution are, $\{2,1\}$ or $\{x : x = 2 \text{ or } x = 1\}$

## Discussion On Factoring.

[In this note we suggest a motivation for trying to factor, and ask whether any polynomial, particularly a quadratic, can always be factored.]

In considering the solution of the equation $ax^2 + bx + c = 0$ it was necessary to rule out the case $a = 0$. [See Remarks on Vocabulary]. When the middle term is missing $b = 0$ the reduced quadratic is linear in $(x^2)$ and its solution reduces to the arithmetic problem of extracting the square root of a number. The case when the constant vanishes, $c = 0$, opens the door to the method of factoring.

For $x = 0$ is an obvious root of $x(ax + b) = 0$, and if $x \neq 0$ the equation demands that $(ax + b) = 0$.

This raises the question: Can we solve a quadratic with $c \neq 0$ by factoring? The answer is "Yes" as set forth in the identity

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

where $x_1$ and $x_2$ are the roots obtained by the quadratic formula.

Thus any quadratic equation which can be solved by completing the square can be solved by factoring, and vice versa.

If any equation $ax^2 + bx + c = 0$ has a root $x = r$, the quadratic function $ax^2 + bx + c$ has the factor $(x - r)$.

For the information that $x = r$ is a root means that $Q(r) = 0$. Then $Q(x) = Q(x) - Q(r)$, subtracting zero, which can be written:

$$\begin{aligned} Q(x) &= (ax^2 + bx + c) - (ar^2 + br + c) \\ &= a(x^2 - r^2) + b\,(x - r) \\ &= (x - r)\,[a\,(x - r) + b]\ , \end{aligned}$$

so that $(x - r)$ is a factor. The proof of the converse is even simpler: if $Q(x)$ factors, $Q(x) = (x - r)\,g(x)$ with $(x - r)$ as one of the factors then any time you substitute $x = r$, the function $Q(x)$ becomes $0 \cdot g(r)$, and this vanishes no matter what $g(r)$ is.

The teacher should observe that this proof applies equally well for cubics and, in fact, polynomials of any degree. For, all binomials of the form $x^s - r^s$ factor with $(x - r)$ as one of the factors.

The one-to-one correspondence between roots and factors is the only logical way of dealing with repeated roots. Once the notion of counting a root $r$ as many times as its factor $(x - r)$ occurs is established, the statement of the fundamental theorem of algebra makes sense. "Every equation of $n^{th}$ degree has exactly $n$ roots."

The proof that an equation of $n^{th}$ degree cannot have more than $n$ roots is contingent on the remarkable property of zero that a product cannot vanish unless at least one of its factors is zero.

The proof of this remarkable property of zero is contingent on the fact that every number except zero has a reciprocal. Students may be interested in an "algebra" where this property does not hold. Consider a "penny" algebra in which the processes of addition and multiplication resemble those of ordinary algebra except that only the last two digits are used. Thus the sum of 83¢ and 40¢ would be 23¢ since we agree to neglect the dollar digit in the sum. In this penny algebra 83¢ and 17¢ would be *negatives* of each other, since their sum is zero. The product of 4¢ and 25¢ would be 0¢, "zero" though *neither of the factors is zero*.

In "penny" algebra the quadratic equation $x^2 = 76$ would have the roots $+24$ and $-24$ corresponding to the factored form $x^2 - 76 = (x - 24)(x + 24) = 0$. But since 2 times 50 is also zero, we might try $x - 24 = 2$ and $x + 24 = 50$, and this gives a third root $x = 26$.

The beautiful "fundamental theorem" does not hold in "penny algebra". Let us return to ordinary algebra.

*Exercises* 4 - 3. *Answers*

1. $\{2,3\}$
2. $\{3,4\}$
3. $\{2,5\}$
4. $\{4,4\}$
5. $\{4,-4\}$
6. $\{13,-13\}$

7. No solution within the domain of real numbers.

8. $\{11, 7\}$

9. The roots are $-3$ and $+2$ .

10. The roots are $5$ and $-2$ .

11. $\{-13, 5\}$

12. $\{9, -6\}$

13. $(7x - 1)(3x + 2) = 0$ .

$7x - 1 = 0$ or $3x + 2 = 0$.

$x = \frac{1}{7}$ or $x = -\frac{2}{3}$ .

The roots are $\frac{1}{7}$ and $-\frac{2}{3}$ .

14. $(7x - 1)(x - 2)$ are the factors.

The roots are $\frac{1}{7}$ and 2.

*15. $11x^2 + 43x - 130 = 0$ is an equivalent equation.

$(11x + 65)(x - 2) = 0$

$11x + 65 = 0$ or $x - 2 = 0$

$x = -\frac{65}{11}$ or $x = 2$

The solution set is $\{-\frac{65}{11}, 2\}$ .

16. $(3x - 2)(x - 4) = 0$

$x = \frac{2}{3}$ or $x = 4$

$\{\frac{2}{3}, 4\}$

17. $(12x - 25)(4x - 21)$ are the factors

$-\frac{25}{12}$ and $\frac{21}{4}$ are the roots.

18. No solution within the domain of real numbers.

19. $(6x - 1)(3x - 1) = 0$

$$6x - 1 = 0 \text{ or } 3x - 1 = 0$$

$$x = \frac{1}{6} \text{ or } x = \frac{1}{3}$$

The roots are $\frac{1}{6}$ and $\frac{1}{3}$ .

20. $\{-5\}$. Here we have "double" or repeated roots.

## 4 - 4. Comments

*Methods for Finding Square roots.* We shall discuss the solution of the *reduced quadratic* in which $b = 0$ so that the "middle term" in $x$ does not occur. The substitution $y = 2ax + b$, or the method of completing the square, reduces *any* quadratic equation to this form. In other words finding "quadratic roots" is based on being able to find "square roots".

The equation $a(x^2) + c = 0$ is quadratic in $x$, but merely linear in $(x^2)$, with the solution

$$x^2 = k,$$

where $k = - c/a$. How do we solve this problem for $k > 0$? If $x_1 = \sqrt{k}$ indicates the positive root, then $x_2 = -\sqrt{k}$ is the other root as can be seen from the factored form $x^2 - k = (x - \sqrt{k})(x + \sqrt{k}) = 0$. How do we take the square root of the positive number $k$? We mention briefly six methods any one of which is within the grasp of an eleventh grade student.

(1) The easiest way to evaluate $\sqrt{k}$ is to look it up in a table of square roots.

(2) If no tables are available many arithmetic texts suggest an algorithm for extracting the square root of any positive number written in the decimal notation.

(3) The rule "divide and average" is a very simple method of finding square roots and leads to a sequence of rational approximations. [See Chapter 1]

(4) A better rational approximation for $\sqrt{N} \approx r$ is $r + (N-r^2)/2r$. From $\sqrt{27} \approx 5$ we are led to consider $5 + \frac{2}{10}$. The error in the square, $N - r^2 = 27 - 25 = 2$, is divided by twice the guessed root for the correction term.

This method is justified by the formula for the square of a binomial. Thus $\sqrt{27} = 5 + x_1$. Squaring, $27 = 25 + 10x_1 + x_1^2$, or $x_1^2 + 10x_1 - 2 = 0$. Assuming that we may nelgect $x_1^2$ since $x_1$ is a proper fraction, and its square is still smaller, we have the linear formula $10x_1 - 2 = 0$, whence $x_1 = .2$. A second and closer approximation is obtainable by beginning with $\sqrt{27} = 5.2 + x_2$; whence $27 = 25 + 2 + .04 + 10.4x_2$, and the second correction is $x_2 = -\frac{.04}{10.4} \approx -.001$.

(5) The $\sqrt{k}$ may be found by logs. (See Chapter 8).

(6) Finally, $\sqrt{12}$ may be written $(9 + 3)^{\frac{1}{2}}$ and expanded by the binomial formula. [See chapter 10.]

Exercises 4 - 4. Answers

1. $\{+4, -4\}$
2. $\{+\frac{12}{13}, -\frac{12}{13}\}$
3. $\{+\sqrt{5}, -\sqrt{5}\}$

4. $\{+\sqrt{101}, -\sqrt{101}\}$
5. $\{+1.7725, -1.7725\}$
6 $\{+\sqrt{5}, -\sqrt[4]{5}\}$ $[\sqrt[4]{5} = \sqrt{\sqrt{5}}]$
7. $\{10, -16\}$
8. The roots are $(4 + \sqrt{7})$ and $(4 - \sqrt{7})$
9. $\{9,1\}$
10. $\{-\frac{3}{2}, \frac{1}{2}\}$

## 4 - 5. Comments

In addition to presenting the method of solving quadratic equations by completing the square it would be advantageous to know why the method works for all quadratics. There is a geometric answer based on the *axis of symmetry* and an algebraic answer based on the *average of the roots*.

Geometrically, we refer to the previous chapter where the graph of a quadratic function, $y = ax^2 + bx + c$ is a parabola *which has an axis of symmetry* at $x = -\frac{b}{2a}$. The binomial $(x + \frac{b}{2a})$ whose square we complete is precisely the displacement measured from this axis. *All parabolas are the same shape* but their equations are simplest when the y-axis coincides with their own symmetry axis. The shift of emphasis from the original abscissa $x$ to the displacement $x + \frac{b}{2a}$ corresponds to a translation of the coordinate axes.

Algebraically, if we take the arithmetic average of the two roots we get

$$h = \frac{1}{2}(x_1 + x_2),$$

which may be compared with the formula for sum of roots (Section 4-9)

$$x_1 + x_2 = -\frac{b}{a}$$

$$h = \frac{1}{2}\left(-\frac{b}{a}\right) = -\frac{b}{2a}.$$

Exercises 4 - 5. Answers

1. $x^2 - 6x = 7$

   $x^2 - 6x + 9 = 9 + y$

   $(x - 3)^2 = 16$

   $x - 3 = \pm 4$

   $x - 3 = 4$ or $x - 3 = -4$

   $x = 7$ or $x = -1$

   The roots are 7 and -1

2. $x^2 - 8x = 180$

   $x^2 - 8x + 16 = 196$

   $x - 4 = \pm 14$

   $x = -18$ or $x = 10$

3. $x^2 - 4x + 4 = 45 + 4$

   $(x - 2)^2 = 49$

   $x - 2 = \pm 4$

   $x = 9$ or $x = -5$

   The solution set is $\{9, -5\}$

4. $x^2 - 2x + 1 = 1 - 7 + 150$

   $(x - 1)^2 = 144$

   $x - 1 = 12$ or $x - 1 = -12$

   $x = 13$ or $x = -11$

   The roots are 13 and -11.

5. $x^2 - 14x + 49 = -45 + 49$

   $(x - 7)^2 = 4$

   $x - 7 = \pm 2$

   $x = 7 \pm 2$

   $x = 9$ or $x = 5$

   The roots are 9 and 5.

6. $x^2 + 5x + \frac{25}{4} = -1 + \frac{25}{4}$

   $\left(x + \frac{5}{2}\right)^2 = \frac{21}{4}$

   $x + \frac{5}{2} = \pm \frac{\sqrt{21}}{2}$

   $x = -\frac{5}{2} \pm \frac{\sqrt{21}}{2} = \frac{-5 \pm \sqrt{21}}{2}$

   The roots are

   $\frac{-5 + \sqrt{21}}{2}$ and $\frac{-5 - \sqrt{21}}{2}$

7) $x^2 + 3x + \frac{9}{4} = 10 + \frac{9}{4}$

$(x + \frac{3}{2})^2 - \frac{49}{4} = 0$

$(x + \frac{3}{2} - \frac{7}{2})(x + \frac{3}{2} + \frac{7}{2}) = 0$

$(x - \frac{4}{2})(x + \frac{10}{2}) = 0$

$(x - 2)(x + 5) = 0$

$x - 2 = 0$ or $x + 5 = 0$

$x = 2$ or $x = -5$

The solution set is $\{2,-5\}$.

8) $x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{9}{16} - \frac{8}{16}$

$(x + \frac{3}{4})^2 - \frac{1}{16} = 0$

$(x + \frac{3}{4} - \frac{1}{4})(x + \frac{3}{4} + \frac{1}{4}) = 0$

$(x + \frac{1}{2})(x + 1) = 0$

$x = -\frac{1}{2}$ or $x = -1$

The roots are $-\frac{1}{2}$ and $-1$.

9) $y^2 + \frac{11}{3}y + \frac{121}{36} = \frac{4}{3} + \frac{121}{36}$

$(y + \frac{11}{6})^2 = \frac{48 + 121}{36}$

$y + \frac{11}{6} = \pm \frac{13}{6}$

$y = \frac{1}{3}$ or $y = -4$

The roots are $\frac{1}{3}$ and $-4$.

10) $x^2 - \frac{5}{6}x + \frac{25}{144} = \frac{25}{144} - \frac{1}{6}$

$(x - \frac{5}{12})^2 \quad \frac{1}{144} = 0$

$x - \frac{5}{12} - \frac{1}{12} = 0$ or $x - \frac{5}{12} + \frac{1}{12} = 0$

$x = \frac{1}{2}$ or $x = \frac{1}{3}$

The solution set is $\{\frac{1}{2}, \frac{1}{3}\}$.

11) $x^2 + 8x + 16 = 12$

$x = -4 \pm 2\sqrt{3}$

The solution set is $\{(-4 + 2\sqrt{3}\}$

12) $t^2 + 5t + \frac{25}{4} = \frac{21}{4}$

$t = (-5 \pm \sqrt{21})/2$

The roots are $\frac{-5 + \sqrt{21}}{2}$ and $\frac{-5 - \sqrt{21}}{2}$

13) $2x^2 + 4x - 1 = 0$

$4x^2 + 8x + 1 = 2 + 1$

$(2x + 1)^2 = 3$

$x = (-1 \pm \sqrt{3})/2.$

The roots are $\frac{-1 + \sqrt{3}}{2}$ and $\frac{-1 - \sqrt{3}}{2}$.

14) $2x^2 + 3x - 1 = 0$

$$(x + \frac{3}{4})^2 = \frac{17}{16}$$

$$x = (-3 \pm \sqrt{17})/4.$$

The roots are $- \frac{1 + \sqrt{17}}{4}$ and $- \frac{1 - \sqrt{17}}{4}$

15) $5x^2 + 2x = 1$

$$(x + \frac{1}{5})^2 = \frac{6}{25}$$

$$x = (-1 \pm \sqrt{6})/5$$

The solution set is $\{- \frac{1 + \sqrt{6}}{5}, \quad - \frac{1 - \sqrt{6}}{5}$

16) $x^2 + 4ax + 4a^2 = 9a^2$

$$x + 2a = \pm 3a$$

$$x = a, \quad \text{or} \quad x = -5a$$

The roots are $a$ and $-5a$.

17) $z^2 + 8nz + 16n^2 = n^2$

$$z + 4n = \pm n$$

$$z = -3n, \text{ or } z = -5n$$

The roots are $-3n$ and $-5n$.

18) $m^2y^2 + 10my + 25 = 49$

$$(my + 5)^2 = 49$$

$$y = - \frac{5 \pm 7}{m}.$$

$$y = \frac{2}{m}, \text{ or } y = \frac{-12}{m}$$

The solution set is $\{\frac{2}{m}, \frac{-12}{m}\}$.

19) $m^2y^2 + 10my + 25 = 1$

$$(my + 5)^2 = 1$$

$$y = -\frac{5 \pm 1}{m}$$

$$y = \frac{-4}{m}, \text{ or } y = \frac{-6}{m}$$

The solution set is $\{\frac{-4}{m}, \frac{-6}{m}\}$ .

20) $x^2 + 2bx = -c$

$$x^2 + 2bx + b^2 = b^2 - c$$

$$x = -b \pm \sqrt{b^2 - c}$$

$$x = -b + \sqrt{b^2 - c} \quad \underline{\text{or}} \quad x = -b - \sqrt{b^2 - c}$$

The roots are $-b + \sqrt{b^2 - c}$ and $-b - \sqrt{b^2 - c}$

## 4 - 6. Comments

In the previous sections we have studied the three methods of solving the quadratic equation; by graphing, by factoring and by completing the square. In this section we show that the method of completing the square leads to a formula that will solve any quadratic equation.

These standard methods are not the only available means of solving quadratic equations. The easiest way to solve any equation is to guess the answer and verify by substitution that it is the correct answer.

## Exercises 4 - 6. Answers

The roots are:

1. 3 and 2

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2. $-10$ and $1$

3. $-\frac{1}{3}$ and $-2\frac{1}{2}$

4. $5$ and $-2\frac{3}{4}$

5. $-2$ and $-\frac{3}{7}$

6. $\frac{-3 + \sqrt{41}}{2}$ and $\frac{-3 - \sqrt{41}}{2}$

7. $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$

8. $-\frac{4}{5}$ and $-1$

9. $\frac{1 + \sqrt{7}}{6}$ and $\frac{1 - \sqrt{7}}{6}$

10. $\frac{3 + \sqrt{41}}{8}$ and $\frac{3 - \sqrt{41}}{8}$

11. $\frac{-7 + \sqrt{53}}{2}$ and $\frac{-7 - \sqrt{53}}{2}$

12. $\frac{-1 + \sqrt{5}}{2}$ and $\frac{-1 - \sqrt{5}}{2}$

13. $\frac{-21 + \sqrt{497}}{14}$ and $\frac{-21 - \sqrt{497}}{14}$

14. $\frac{1}{4}$ and $-\frac{1}{3}$

15. $\frac{-11 + \sqrt{145}}{4}$ and $\frac{-11 - \sqrt{145}}{4}$

16. $\frac{-b + \sqrt{b^2 + 4ac}}{c}$ and $\frac{-b - \sqrt{b^2 + 4ac}}{c}$

17. $\frac{ab + b\sqrt{a^2 - 4a - 4b}}{2(a + b)}$ and $\frac{ab - b\sqrt{a^2 - 4a - 4b}}{2(a + b)}$

18. $\frac{-rs + \sqrt{r^2s^2 - 4r^2 + 4rs}}{2r}$ and $\frac{-rs - \sqrt{r^2s^2 - 4r^2 + 4rs}}{2r}$

19. $\frac{1}{\sqrt{E_0}}$ a double root.

20. 2 and $\frac{3}{2}$.

4 - 7. Comments

In the previous sections of this chapter, our motivation was learning to solve quadratic equations. In this and succeeding sections the emphasis is on understanding the intrinsic structure of these equations. The notion of the discriminant will reappear in many branches of higher mathematics.

Tangency and the Intersections of Graphs. Sometimes it is more important to predict the nature of the roots than it is actually to solve for them. Below are some examples.

First Example: Find the second intersection of the circle $x^2 + y^2 = 5$ with the line $y - 1 = m(x - 2)$ if its slope $m$ is rational. Show that its coordinates are also rational.

We note that when the coefficients $a,b,c$ are rational, if one root is rational the second root is also rational. One intersection is the point $(2,1)$. On eliminating $y$ the quadratic for $x$ will have rational coefficients. Since one root $x = 1$ is rational, so is the other. A similar argument holds if you eliminate $x$. Therefore, the coordinates of the second intersection are rational.

Second Example: Find the equation of a line tangent to $y^2 = 12x$ at $x = 3$, $y = 4$. The equation of the line is $y - 4 = m(x - 3)$ providing we have the right value for the slope $m$. When the line is solved simultaneously with the parabola it cannot have distinct roots. Therefore, the discriminant must be zero, which determines m.

Exercises 4 - 7. Answers.

In order to compare the value of the discriminant and the graph, have the students graph the first three equations on the same pair of axes.

1. The value of the discriminant is $> 0$; the graph cuts the x-axis in two places (6,0) and (2,0).
2. The value of the discriminant is equal to 0; the graph is tangent to the x-axis at (4,0).
3. The value of the discriminant is $< 0$; the graph does not intersect or touch the x-axis.
4. (a) One element; the value of the discriminant is 0.
   (b) Two elements; the value of the discriminant is $> 0$.
   (c) No elements; the value of the discriminant is $< 0$.
   (d) No elements; the value of the discriminant is $< 0$.
   (e) Two elements; the value of the discriminant is $> 0$.
   (f) No elements; the value of the discriminant is $< 0$.

## 4 - 8. Comments

This section is more or less self explanatory. Example 5 and exercise 9 should show the student the value of finding the equation when the roots are known.

Exercises 4 - 8. Answers.

1) $x^2 - 11x + 30 = 0$

2) $x^2 + 4x - 21 = 0$

3) $x^2 - 8x + 16 = 0$

4) $x^2 - 36 = 0$

5) $x^2 = 0$

6) $3x^2 - 11x - 4 = 0$

7) $5x^2 - 2x = 0$

8) $12x^2 + x - 6 = 0$

9) See Example 5 in textbook. $y = -2(x - 2)(x - 8)$.

10) $x_1x^2 - (x_1^2 + 1)x + x = 0$

## 4 - 9. Comments

This is a companion section to Section 4 - 8 and it shows that every quadratic equation is factorable.

Exercises 4 - 9. Answers

1) Sum is 13; product is 40

2) Sum is -5; product is - 50

3) Sum is 3; product is $2\frac{1}{2}$ or $\frac{5}{2}$

4) Sum is 3; product is 0

5) Sum is $\frac{11}{7}$; product is $-\frac{8}{7}$

6) Sum is ab; product is a + b

7) Length is x; width is $10 - x$; Area is $x(10 - x) = 21$

$$10x - x^2 = 21$$

$$x^2 - 10x + 21 = 0$$

Sum is 10; product is 21.

8) Sum is 4; product is 1

$$x^2 - 4x + 1 = 0$$

9) One root is $x_1 = 5 + \sqrt{11}$.

Suppose the root is $x_2$

The sum of the roots is 10 $\therefore$ $(5 + \sqrt{11}) + x_2 = 10$,

and $x_2 = 5 - \sqrt{11}$.

10) $\frac{-4 - \sqrt{5}}{3}$ is the other root.

Sum is $-\frac{8}{3}$; product $= \frac{11}{9}$

$$x^2 + \frac{8}{3}x + \frac{11}{9} = 0$$

$$9x^2 + 24x + 11 = 0 .$$

## 4 - 10. Comments

This section along with 4 - 11 and 4 - 12 do not necessarily belong with quadratic equations. But since they tend to broaden the students'ability to solve equations of various types, they are included at this point. A summary of the necessary precautions which one must observe is given just preceding 4 - 11 in the text.

## 4 - 10 Oral Exercises Answers

1. Equivalent.
2. Equivalent.
3. B is deficient since 0 is lost.
4. B {0,-1}, but -1 is extraneous for A.

5. A $\{x : x^4 - 4x^2 = 0 \text{ or } x^4 + 4x^2 = 0\}$,

   B $\{x : x^4 - 4x^2 = 0\}$,

   If $x$ is real, then $\{x : x^4 + 4x^2 = 0\} = \{0\}$,

   Therefore $A \equiv B$.

6. B is deficient since 0 is lost.
7. Equivalent.
8. Equivalent.
9. $A \equiv \{2,3\}$, $B \equiv \{1,-1\}$ No equivalent relation.
10. Equivalent.

### Exercises 4 - 10 Answers

Emphasize to the student that in each case we exclude from the domain of definition those values for $x$ which make the denominator vanish. When students give the solution insist that this restriction be pointed out. For example:

$$x - \frac{4}{x} + 3 = 0$$

$$x\left(x - \frac{4}{x} + 3\right) = x(0) \quad [x \neq 0]$$

The roots are:

1) -4,1
2) +3, -3
3) -1, -1
4) 1, 1
5) 0, 2
6) -2, 1
7) -7,5

8) 0, 17

9) 2, since 5 is not in the domain

10) No solution.

### Exercises 4 - 11 Answers

1) {2}

2) {3, -1}

3) {7} (4 is extraneous)

4) {5} ( $\frac{8}{3}$ is extraneous)

5) {10}

6) {+ 3, -3}

7) {1} ($\frac{1}{4}$ is extraneous)

8) $\{\frac{1}{2}, \frac{1}{18}\}$

9) {2} (134 is extraneous)

10) $\{-1, \frac{1}{3}\}$

11) {1, 3}

12) {3} (7 is extraneous.)

### Exercises 4 - 12 Answers

1) $\{\sqrt{3}, -\sqrt{3}, 1, -1\}$

2) $\{\sqrt{2}, -\sqrt{2}, 2, -2\}$

3) {3, -3, 2, -2}

4) {5, -5, 2, -2}

5) {2, -3} (There are only two real roots.)

6) {2, 1} (There are only two real roots.)

7) {4, -1, 2, 1}

8) $\{-2, -1, \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}\}$

9) $\{\frac{17}{9}, \frac{9}{4}\}$

10) $\{2\}$

Many of these equations (such as 1-4) could have been solved by factoring instead of substitution. The teacher can use his own discretion as to which way he wants the student to solve these.

### 4 - 13 Comments

One must be aware that mathematics is more than a game played with a system of elements [the real numbers up to now] according to a set of rules [for addition and multiplication]. It provides a means for solving problems which arise in science and in the practical affairs of the world. The student needs practice in translating a "word problem" into algebraic symbolism. Even a machine can turn a crank on a formula, but it takes judgment to know how to tackle a problematic situation. One should inform the student that it is not at all unusual to spend more time in formulating the equation than in completing the solution.

In all problems one needs to pay careful attention to the precision of the data and their relevance to the problem, basic principles, and checking the answer in the initial equation [for computational errors] as well as in the problem situation. The following illustrative problems include some additional opportunities for mathematical growth. These opportunities will be suggested in the comments following the solution of each problem.

Problem 1.

The length of a rectangle is 3 feet longer than its width. The area is 40 ft.$^2$ What are its dimensions?

Solution:

Let $x$ be the width (in ft.). Then $(x + 3)$ is the number of feet in the length. The variable $x$ must satisfy the equation

$$x(x + 3) = 40.$$

It is easy to see that the expression $x(x + 3)$ is exactly 40 if $x = 5$, it is more than 40 if $x > 5$, and less than 40 if $x < 5$, Accordingly the width $x$ of the rectangle must be 5, and the length $(x + 3)$ is 8.

Problem 2.

Cuneiform tablets dating from around 1200 B.C. indicate that the Babylonians could solve problems of this sort. A modern version might read: "The difference of two integers is 3, their product is 40. What are these numbers?"

Solution:

The equation $f(x) = x^2 + 3x - 40 = 0$ can be written in the factored form

$$f(x) = (x - 5)(x + 8) = 0$$

The factored form leads us quickly to two roots: $x_1$ such that $(x_1 - 5) = 0$, and $x_2$ such that $(x_2 + 8) = 0$.

But the method of factoring also shows that no third root $x_3$ is possible. For if $x_3 \neq 5$ and $x_3 \neq -8$, then $(x_3 - 5)(x_3 + 8)$ cannot be zero.

The root -8 was extraneous for the original problem but both 5 and -8 are acceptable for the modern version, Frequently a slight rewording of the original problem will convert "extraneous roots" into usable answers. This feature is also found in our next example.

Problem 3.

A boy retrieving a home run throws a baseball with an upward initial speed of 60 ft./sec. in order to clear a 44 ft. wall. At what time will the ball be 44 ft. above the level from which it was thrown?

Solution:

Ignoring gravity the initial speed would carry the ball to a height $s_1 = 60t$ feet in $t$ seconds. During the same time a ball starting from rest would fall $s_2 = 16t^2$ feet. Physics tells us that these two motions act independently, so that the net rise can be obtained by their algebraic superposition, $s_1 - s_2 = 44$. This yields the equation

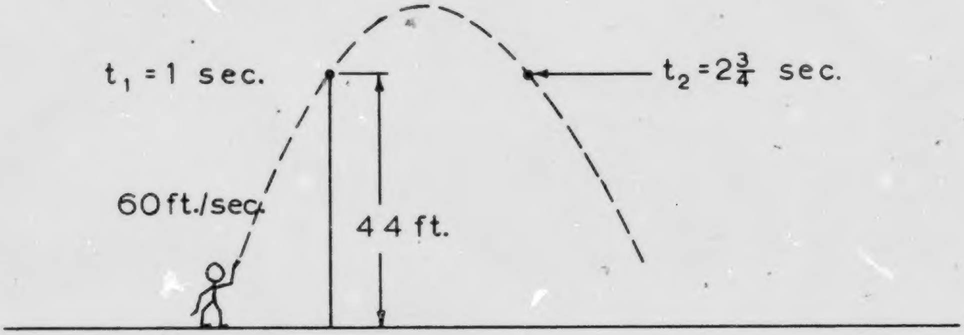
$$60t - 16t^2 = 44$$

which may be written

$$4t^2 - 15t + 11 = 0$$

If we happen to notice that the sum of the coefficients, $4 - 15 + 11$, vanishes, we know that $t = 1$ is certainly a root. This suggests the factor $(t - 1)$ and indeed the factors of

($4t^2 - 15t + 11$) are ($t - 1$) and ($4t - 11$). The second factor provides a second root $t = \frac{11}{4}$. The ball will be just high enough to clear the fence after 1 second (on the way up) and again after $2\frac{3}{4}$ seconds (on the way down). Both roots seem to be relevant to the actual situation from which the algebraic equation was derived.



Comments

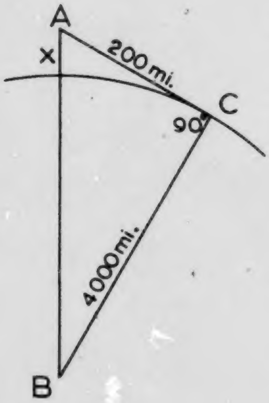
One should not be too much in a hurry to discard these alternate "solutions". In modern physics occurs the equation $E^2 = m^2c^4$ which is a quadratic equation for the energy $E$ of an elementary particle. The root $E = mc^2$ was recognized as the "correct" solution, being Einstein's formula where $m$ is the mass of an electron and $c$ is the speed of light. Dirac joked about a particle of negative mass corresponding to the second root $E = -mc^2$, but Oppenheimer discovered the "positron" which, having an electric charge opposite to that of the electron, is just sufficient to account for the minus sign. Perhaps even negative mass ($-m$) or speeds whose squares are negative will find interpretations in nuclear physics.

This problem has many teaching values. Of paramount importance is the necessity of knowing the scientific principle which governs the relation among the relevant physical quantities. Units must be carefully assigned. The speed 60 ft./sec. may be thought of as a quotient "60 feet divided by 1 second".

The solution stresses the merit of a lucky guess. The teacher is doing a disservice to mathematics if he insists on a laborious approach when anyone can "see" the answer. Factoring takes advantage of such "obvious roots" and shows how this makes the finding of the other, not-so-obvious, roots easier.

Problem 4.

A balloon is $x$ miles above sea level. Regarding the earth as a sphere of radius 4000 miles and ignoring the refraction of light, if the horizon is 200 miles distant along the line of light, what is the altitude of the balloon?



Solution:

Line of sight AC from helicopter at A to horizon at C is perpendicular to radius BC, By the Pythagorean theorem

$$(4000)^2 + (200)^2 = (4000 + x)^2$$

Taking the positive square root of each side

$$200\sqrt{401} = 4000 + x$$

so that $x = 200(\sqrt{401} - 20) = 4.997$

or approximately 5 miles.

Comments:

How many digits in the square root of 401 are needed for 4-place division in the answer? The problem teaches the value of carrying radicals through the computation, and only introducing rational approximations when we are evaluating the answer. The Pythagorean theorem and proportions from similar triangles are constantly recurring themes in reducing geometrical problem situations to algebraic equations.

In abstracting the mathematical formulation and interpreting the answer the student must not only take into account that the earth is slightly pear shaped, but also that light does not travel through the atmosphere in straight lines. A mathematical answer cannot be more accurate than the relevance of its simplifying assumptions.

In selecting problems for blackboard discussinn teachers usually have a preference for equations whose coefficients are simple integers. It should be born in mind, however, that in the application of mathematics this is the exception rather than the rule. This problem shows how unusually large numbers may arise and the important notion of precision is introduced. Does 4000 miles mean "to the nearest mile," 4000 x 5280 ft., or what? If the mathematical solution could be carried out to any desired accuracy how accurate is the data? Note that in the computation itself a surprising number of decimal places for $\sqrt{401}$ must be computed to get 4-place precision in the root. One must make a careful distinction among "*errors* in measurement" (which indicate the precision of our data), "*inaccuracies* in computation" which can usually be reduced, and "*mistakes* in applying the fundamental rules."

*Problem* 5.

A swimmer can cross to the opposite side of a river of width $w$ feet and current $c$ feet per second and return in the same time as it would take him to swim downstream $d$ feet and return. How fast does he swim?

*Solution*:

If $x$ is the speed of the swimmer then

$$\frac{2w}{\sqrt{x^2 - c^2}} = \frac{d}{x - c} + \frac{d}{x + c} = \frac{2dx}{x^2 - c^2}$$

Squaring

$$\frac{w^2}{x^2 - c^2} = \frac{d^2x^2}{(x^2 - c^2)^2}$$

so that $w^2(x^2 - c^2) = d^2x^2$

Although this equation with literal coefficients is quadratic in $x$, it is merely linear in $(x^2)$. For, $(w^2 - d^2)x^2 = w^2c^2$ , so that $x^2$ has the unique value $w^2c^2/(w^2 - d^2)$ and accordingly

$$x = \frac{wc}{\sqrt{x^2 - d^2}}$$

Comments:

It is best to resist the temptation to rationalize the denominator. This formula tells us that there cannot be a real answer unless d is less than w. The contrast between the average speeds across the current, $2dx/(x^2 - c^2)$, and parallel to the current, $2w/\sqrt{x^2 - c^2}$, was used by physicists at the turn of the century to measure the velocity of the ether relative to the earth. Since no difference was detectable although the earth reversed its orbital velocity twice a year, this quadratic equation led scientists to doubt the physical reality of ether. This problem leads to an equation with literal coefficients. All the drill on factoring trinomials with integral coefficients is no help here. The student must have confidence in operating with such numbers (often called parameters) as well as with constants which are "known" in the way we know the counting integers.

Exercises 4 - 13 Answers

1) 3" is the width of the frame

2) 15 m.p.h. is rate downstream; 9 m.p.h. is rate upstream

3) 20, 21

4) a) 2 seconds b) 10 seconds c) S(8) - S(7), S(10) - S(9).

5) $\Delta = 0$, $k = 36$

6) a) $K = \pm 2$, b) $K = -8$.

7) $e_1 = 12$; $e_2 = 10$

8) a) 24 oranges per bag; b) \$.50 per dozen.

9) Solution: Let $v$ be the average speed of the first train. (Note that $v$ is a number, as yet unknown to us.) Then $v + 5$ is the average speed of the second train. By making use of the distance formula $d = vt$ we can express the time required by the first engine to pull the train 140 miles as $\frac{140}{v}$. The time required by the second engine to pull the train 200 miles is $\frac{200}{v+5}$. Since the total time required is 9 hours, we have the equation

$$\frac{140}{v} + \frac{200}{v+5} = 9.$$

Multiply both members of this equation by $v(v + 5)$; we have

$$140(v + 5) + 200v = 9(v^2 + 5v),$$

or

$$9v^2 - 295v - 700 = 0$$

Solving by the quadratic formula:

$$v = \frac{295 \pm \sqrt{112,225}}{18},$$

$$= \frac{295 \pm 335}{18},$$

$$= 35, -\frac{20}{9}.$$

The answer $-\frac{20}{9}$ has no meaning for our problem. The average speed of the first train is 35 miles per hour, and the average speed of the second train is 40 miles per hour.

10) The volume of a cone is $V = \frac{1}{3}\pi(100 - x^2)x$ where $x$ is the altitude and 10 is the slant height. Since this is a cubic the quadratic formula will not work. If the

student constructs a table of values for the function $V(x)$, or plots its graph he will find that a maximum for $V$ occurs between $x = 5$ and $x = 6$ inches. [It is easier to plot $y = (100 - x^2)\,x$.]

### *4 - 14 Comments

The reason that this section is starred is not due to its difficulty, but because it will be treated more fully in grade 12.

### Exercises 4 - 14 Answers

1) $\{3, 7, -4\}$
2) $\{8, -11, \frac{1}{2}\}$
3) $\{\frac{3}{2}, -\frac{1}{3}, -\frac{1}{5}\}$
4) $R = 0$
5) $R = (-1)^{63} + (-1)^{17} + 5 = 3$

### 4 - 15 Comments

This material on inequalities has been included in grade eleven to correct the faulty notion that algebra deals only with the solution of equations. The treatment in this section, however, makes much use of two methods that correspond to those used in solving quadratic equations: completing the square and factoring.

You will need to review the transformations for inequalities. You will find the following material in Chapter I useful: D. 14, (PO 4 or T 26),(PO 5 or T 30 or T 31), and number 3, Exercises 1-9d.

Insist that the student graph the inequalities. This will help him have a better notion of their solution. Set notation is very helpful in describing the solution set of inequalities. It may or may not be stressed. However, it is recommended that you point out to your students the use of this notation.

Some of the exercises in this set are difficult, for example, number 5 and number 18. Use your discretion in selecting particular ones for your pupils. You may wish to precede these exercises with simpler ones, such as,

(1) $|x| > x$
(2) $|x| < x$
(3) $|x| > x^2$
(4) $|x| < x^2$
(5) $(|x| + 5) > x$
(6) $|x| > x + 5$
(7) $|2x - 1| > x^2$
(8) $|2x - 1| < x^2$

Beginning with number 12, the students are graphing on the number plane. The graphs will be regions. You may wish to begin this group with simpler inequations such as,

(1) $y < x^2$
(2) $y < |x|$
(3) $y > x^2 + 1$
(4) $y \geq x^2 + 1$
(5) $y < x^2 + 1$
(6) $y > |x^2|$
(7) $y < |x^2|$
(8) $y^2 < x$

4 - 15 Exercises Answers

1) $x^2 - 4x + 3 < 0$

2) $(x + 1)(x - 3) < 0$

Since this product is negative,

$[x + 1 < 0$ and $x - 3 > 0]$ or $[x + 1 > 0$ and $x - 3 < 0]$

Chapter I, Ex. 1-9d number 3.

$\therefore$ $[x < -1 \text{ and } x > 3]$ or $[x > -1 \text{ and } x < 3]$ Chapter I PO 4 or T 26, which is,

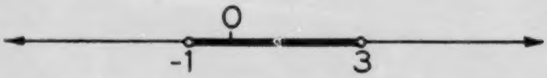
$$[\quad \emptyset \quad] \text{ or } [-1 < x < 3]$$

This union is,

$$[-1 < x < 3]$$

The solution set is, $\{x \mid -1 < x < 3\}$

Its graph is,



2) $x^2 + 4 \geq 5x$

$x^2 - 5x + 4 \geq 0$

$(x - 4)(x - 1) \geq 0$

The symbol, $\geq$, means " is equal to or is greater than."

Hence, we write,

$[(x - 4)(x - 1) > 0]$ or $[(x - 4)(x - 1) = 0]$

$[(x - 4 > 0 \text{ and } x - 1 > 0)$ or $(x - 4 < 0 \text{ and } x - 1 < 0]$ or

$[x - 4 = 0 \text{ or } x - 1 = 0]$

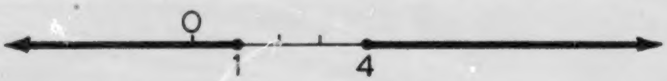
$[(x > 4 \text{ and } x > 1)$ or $(x < 4 \text{ and } x < 1)]$ or

$[x = 4 \text{ or } x = 1]$

$[(x > 4) \text{ or } (x < 1)]$ or $[x = 4 \text{ or } x = 1]$

The solution set is, $\{x \mid [x > 4 \text{ or } x < 1] \text{ or } [x = 4 \text{ or } x = 1]\}$.

Its graph is,



Hence, the solution set has a graph which is the union of two half lines, $x \leq 1$ and $x \geq 4$.

3) $x^2 - x < -1$

Completing the square,

$x^2 - x + \frac{1}{4} < -\frac{3}{4}$

$(x - \frac{1}{2})^2 < -\frac{3}{4}$

Since $(x - \frac{1}{2})^2$ must be negative to be less than $- \frac{3}{4}$, which is impossible, the solution set is empty. Some authors use the symbol, $\emptyset$, to indicate the empty set.

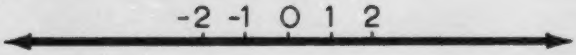
4) $x^2 - x > 1$

$(x - \frac{1}{2})^2 > - \frac{3}{4}$, by completing the square.

Since any number squared is always $> 0$, then $(x - \frac{1}{2})^2$ will be equal to 0 or be positive for any real numbers. Hence, for any real number for x, $(x - \frac{1}{2})^2 > - \frac{3}{4}$.

The solution set is every real number. Some students may like to invent ways of describing this in set notation.

Its graph is the entire number line.



5) (This is difficult)

$$x^2 < |x| + 5$$

Substitute a, for $|x|$ and $a^2$ for $x^2$

$$a^2 < a + 5$$

$$a^2 - a + \frac{1}{4} < \frac{21}{4}$$ , completing the square.

$$a - \frac{1}{2} < \frac{1}{2}\sqrt{21}$$

$$a < \frac{1 + \sqrt{21}}{2}$$

$$\therefore \ |x| = a < \frac{1 + \sqrt{21}}{2}$$

By definition this is,

$$- x < \frac{1 + \sqrt{21}}{2} \quad \text{and} \quad x < \frac{1 + \sqrt{21}}{2}$$

$$x > - \frac{1 + \sqrt{21}}{2} \quad \text{and} \quad x < \frac{1 + \sqrt{21}}{2}$$ Chapter I, P05

$\frac{1+\sqrt{21}}{2} > x > -\frac{1+\sqrt{2}}{2}$ and x lies within this interval.

Another method:

1) Consider the case, $x > 0$, then $|x| = x$ we have,

$x^2 - x + \frac{1}{4} < \frac{21}{4}$

$(x - \frac{1}{2})^2 = |x - \frac{1}{2}|^2 < \frac{21}{4}$ The use of | | here is a clever idea as it eliminates the necessity of reversing < when the negative root is considered.

$|x - \frac{1}{2}| < \frac{\sqrt{21}}{2}$

Hence x seems to lie in the interval,

$\frac{1}{2} - \frac{1}{2}\sqrt{21} < x < \frac{1}{2} + \frac{1}{2}\sqrt{21}$

But , since we are considering the case, $x > 0$, then x really lies in the interval, $0 < x < \frac{1}{2} + \frac{1}{2}\sqrt{21}$.

2) Consider the case, $x < 0$, then $|x| = -x$ [Note, the absolute value of x is actually a function of x]

So $x^2 < |x| + 5$ becomes

$x^2 + x + \frac{1}{4} < \frac{21}{4}$

$(x + \frac{1}{2})^2 < \frac{21}{4}$

$|x + \frac{1}{2}| < \frac{1}{2}\sqrt{21}$

It seems that x lies within the interval

$-\frac{1}{2} - \frac{1}{2}\sqrt{21} < x < -\frac{1}{2} + \frac{1}{2}\sqrt{21}$

But, x actually lies within the interval,

$-\frac{1}{2} - \frac{1}{2}\sqrt{21} < x < 0$ for $x < 0$

3) Consider the case $x = 0$, Substituting 0 for x we find that 0 satisfies the inequality,

Hence, the solution set is, $\{x: -\frac{1}{2} - \frac{1}{2}\sqrt{21} < x < \frac{1}{2} + \frac{1}{2}\sqrt{21}\}$

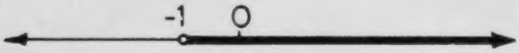
6) $x > \frac{1}{x}$ [$x \neq 0$, since $x = 0$ is not in the domain of definition].

(1) If $x > 0$, then $x^2 > 1$ and $x > 1$

(2) If $x < 0$, then $x^2 > 1$ and $x > 1$

Hence, the solution set $\{x: [x \neq 0]$ and $x > -1\}$

The graph is,



7) $x^2 > 8 > x$

This means, $x^2 > 8$ and $8 > x$

$$\left[\begin{array}{l} x^2 > 8, \text{ if x is positive and } x > \sqrt{8} \\ \qquad \text{or if x is negative and } x - \sqrt{8} \end{array}\right] \text{ and } x < 8$$

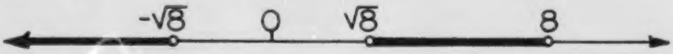
So [$x > \sqrt{8}$ or $x < -\sqrt{8}$] and $x < 8$

[$x > \sqrt{8}$ and $x < 8$] or [$x < -\sqrt{8}$ and $x < 8$]

$\sqrt{8} < x < 8$ and $x < -\sqrt{8}$

Hence, the solution set consists of the negative numbers on the half line, $x < -8$ and the positive numbers in the interval $\sqrt{8} < x < 8$.

Its graph is,



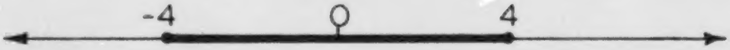
8) $x^2 - 16 \leq 0$

Using the symbol, | |, as in number 5, this equation is equivalent to $|x| \leq 4$

Hence $-4 \leq x \leq 4$

Another method is to factor the left member.

The graph is,

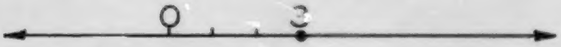


9) $(x - 3)^2 \leq 0$

Since $(x - 3)^2$ can never be negative for any x, then $x = 3$ is the only solution.

The solution set is $\{3\}$

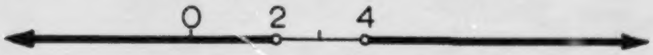
Its graph is,



10) $(x - 2)(x - 4) > 0$

If both factors are positive, then $x > 2$ and $x > 4$ which is $x > 4$. If both factors are negative, then $x < 2$ and $x < 4$ which is $x < 2$. Hence, the union of these two half lines is the solution set, $\{x : x < 2 \text{ or } x > 4\}$

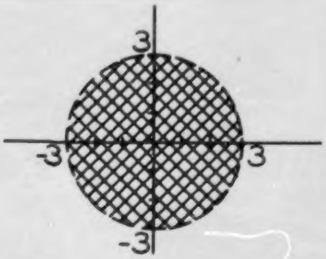
Its graph is,



11) $x^2 + y^2 < 9$

Consider the graph of $x^2 + y^2 = 9$.

It is a circle with center at $(0,0)$ and radius 3.

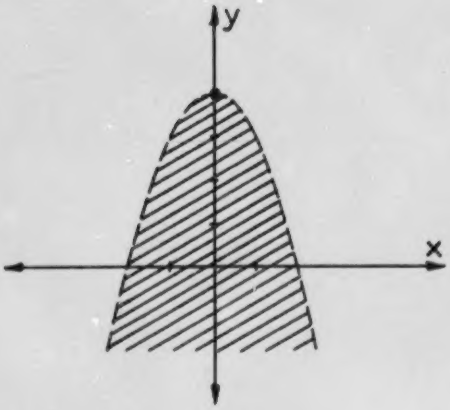


Now, the values for x and y that satisfy $x^2 + y^2 < 9$ is, $\{(x,y) : x^2 + y^2 < 9\}$

Its graph are the points within the circle above, but not including the circle, hence the dotted line. You may wish to have your students construct a table of values that satisfy $x^2 + y^2 = 9$, then locate points which satisfy $x^2 + y^2 < 9$.

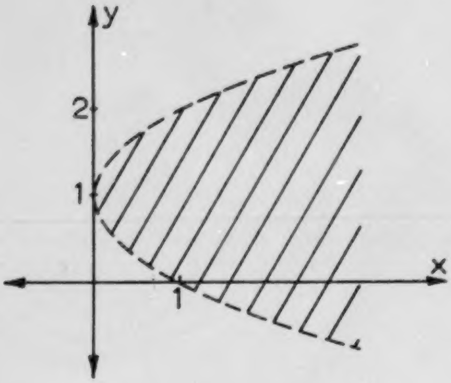
12) $y < 4 - x^2$

Graph the parabola $y = 4 - x^2$. The points $(x,y)$ satisfying the inequality must be below this curve so the points are those inside (but not on) the parabola. Have students check to see if ordered pairs such as, (1, -2), $(\frac{1}{2}, 5)$, ( -1,5), (-1, -3), (5,0),(-6,4) etc.



13) $(y - 1)^2 < x$

Graph the points satisfying the inequality lie to the right of this curve, and are therefore inside (but not on) the parabola.



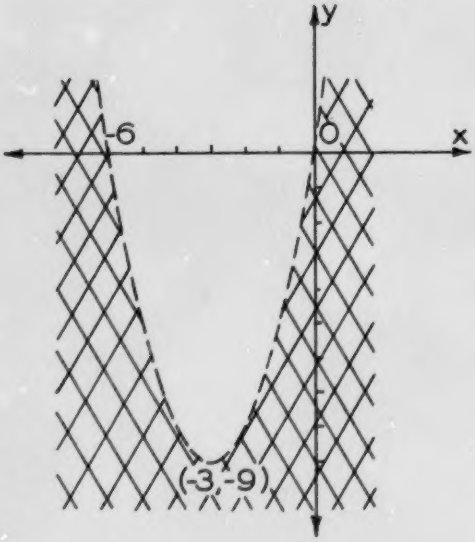
14) $x^2 + 6x > y$

$(x^2 + 6x + 9) - 9 > y$

$(x + 3)^2 - 9 > y$

Graph $(x + 3)^2 - 9 = y$

Since the ordinate of y must be less than this graph, the graph of solution set is the outside region.



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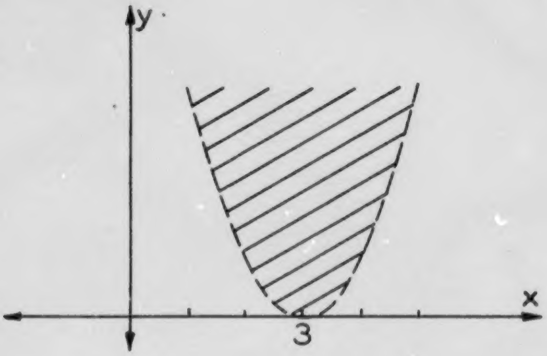
15) $y > x^2 - 6x + 9$

Graph $y = (x - 3)^2$

Since y is greater than $(x - 3)^2$, the graph of the solution set is the shaded region.

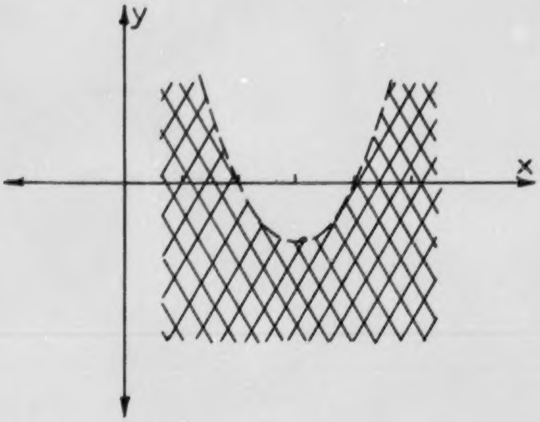


16) $y < x^2 - 6x + 8$

The graph of the solution set is the shaded region outside the parabola.

Ask the students to graph

$y > x^2 - 6x + 8$.



17) $x^2 < y^2$ is equivalent to

$(x - y)(x + y) < 0$

Hence,

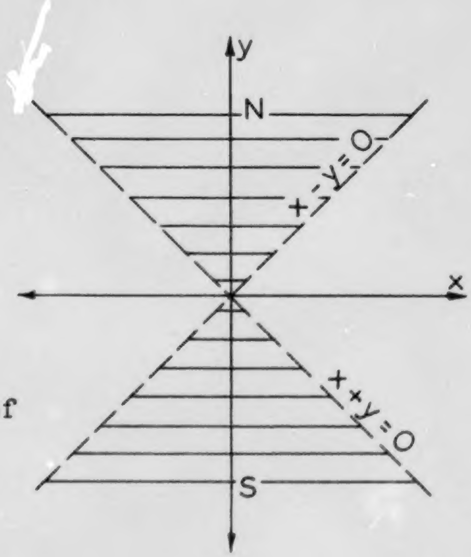
$[(x - y) < 0$ and $(x + y) > 0]$

or $[(x - y) > 0$ and $(x + y) < 0]$

$[x < y$ and $x > - y]$ or

$[x > y$ and $x < - y]$ The graph of $x^2 = y^2$ is two straight lines.

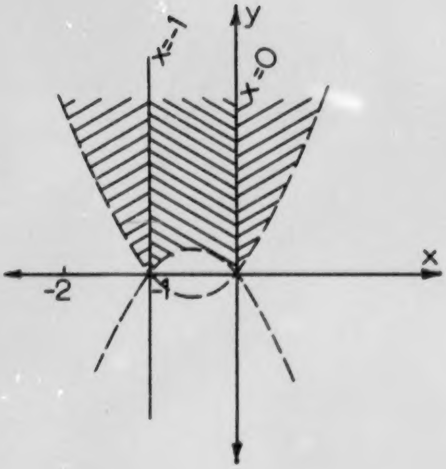
The graph of the solution set of $x^2 < y^2$ is the shaded region.



Region N is the graph of $(x - y) < 0$ and $(x + y) > 0$; while region S is the graph of $(x - y) > 0$ and $(x + y) < 0$.

Hence, the graph of the solution set of $x^2 < y^2$ is the union of these two graphs.

18) $|x^2 + x| < y$



Graph $y = |x^2 + x|$

Other than those values where $x^2 + x = 0$, consider $x^2 + x > 0$ or $x^2 + x < 0$

(1) If $x^2 + x > 0$, then

$|x^2 + x| - x^2 + x$

So, the graph of $y = x^2 + x$ is the parabola. Hence, the graph of $x^2 + x < y$ is above this parabola and in the domain $x^2 + x > 0$, i.e. $x > 0$ and $x < - 1$. For, $x(x + 1) > 0$, then $[x > 0$ and $( + 1) > 0]$ or $[x < 0$ and $x + 1 < 0]$ $[x > 0 ]$ or $[x < - 1]$

(2) If $x^2 + x < 0$, then

$x(x + 1) < 0$, or

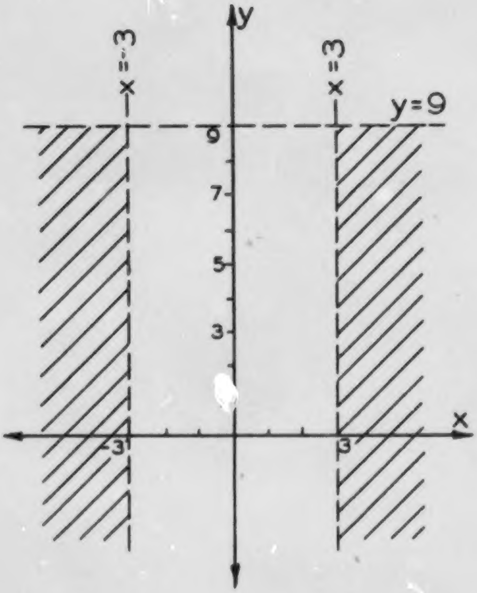
$[x < 0$ and $(x + 1) > 0]$ or

$[x > 0$ and $(x + 1) < 0]$

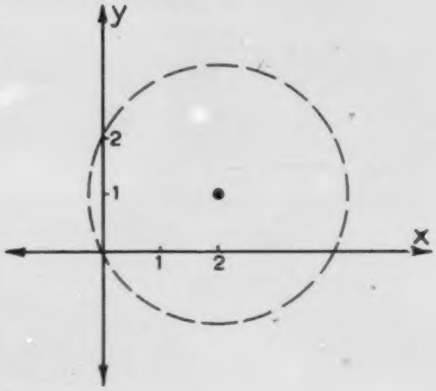
$[-1 < x < 0 ]$ or [ Empty ]

The union gives the interval, $[ -1 < x < 0]$ So, the graph of the solution set of $-(x^2 + x) < y$ in the domain $x^2 + x < 0$ is the region above the parabola and between $x = - 1$ and $x = 0$. $\therefore$ The graph of the solution set of $|x^2 + x| < y$ is the union of these two regions and $x = - 1$, $x = 0$.

19) $x^2 > 9 > y$ may be written $x^2 > 9$ and $9 > y$. The graph of $x^2 = 9$ is the union of the graph of $x = 3$ and $x = -3$. The graph of $y = 9$ is shown. The graph of $x^2 > 9$ is the union of the graphs of $x > 3$ and $x > -3$ is the region on either side of the lines $x = 3$ and $x = -3$ But, to satisfy $9 > y$ also, the graph must exclude the region above the line $y = 9$. Suggest that students consider $x^2 > 9 > y$ and $x^2 > 9 > y$.



20) $(x - 2)^2 + (y - 1)^2 < 5$
Center of the circle is at (2,1). Radius measure is $\sqrt{5}$ The graph is the interior region. Suggest graphing $(x - 2)^2 + (y - 1)^2 = 5$ and $(x - 2)^2 + (y - 1)^2 > 5$.



Suggested test questions.

1. Write the solution set of the equation $(x - 1)(x + 2) = 4$.
2. Which of the following equations are equivalent to the equation $2x - 1 = 0$?

(a) $x = \frac{1}{2}$

(b) $4x^2 - 1 = 0$

(c) $4x^2 - 4x + 1 = 0$

(d) $12x^2 - 4x - 1 = 0$

3. Find the value of $k$ for which the equation $x^2 - 3kx + k = 2$ has $x = 4$ as a root.

4. Find the zeros of the function $y = x^2 - 100x + 2499$.

5. Solve for all real values of x: $2x^4 - 17x^2 - 9 = 0$.

6. Simplify $a\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^2 + b\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + c$.

7. Derive a formula for the solutions of the equation $cx^2 + ax + b = 0$.

8. Solve for all real values of x: $\sqrt{2x + 1} = 2x - 5$.

9. Sketch the graph of $y = 3 + x - 2x^2$, showing its intercepts and its general shape.

10. Find the range of values of $k$ for which the equation $kx^2 - 2x + 3 = 0$ has two distinct real roots.

11. Find a quadratic equation with integral coefficients whose roots are $3$ and $-\frac{2}{5}$.

12. Find a quadratic equation whose roots are the negative reciprocals of the roots of $6x^2 + 11x - 10 = 0$.

13. Find the sum and the product of the roots of the equation $3x^2 + 5x - 7 = 0$.

14. Find the value of $k$ for which the roots of the equation $2kx^2 - 3k^2x + 5k - 2 = 0$ are the reciprocals of each other.

15. Find the solution set of the equation $\frac{2x^2}{x + 1} + 1 = \frac{2}{x + 1}$.

16. Solve the equation $\sqrt{2x + 5} - \sqrt{x - 1} = 4$.

17. Find the solution set of the equation
$(x^2 - 3x + 1)^2 - 4(x^2 - 3x + 1) - 5 = 0$.

18. The distance S in feet that a body starting from rest falls in t seconds is given by $S = 16t^2$. If a stone falls from the top of a 60-story building and takes 2 seconds to cover the bottom 30 stories, how high is the building?

19. Find the solution set of the following sentence in the number system of integers:

$$-4 < 2x + 3 < 12.$$

20. Which of the following are identities for the set mentioned?

(a) $(x - 1)^2 = x^2 - 2x + 1$ $\{x : x \text{ a real number}\}$

(b) $\frac{x^2 - 1}{x - 1} = x + 1$ $\{x : x = 1, 2, 3, \text{ or } 4\}$

(c) $x^3 - 6x^2 + 11x - 6 = 0$ $\{x : x = 1, 2, \text{ or } 3\}$

(d) $x + \frac{1}{x} > 2$ $\{x : x \text{ a positive integer}\}$

21. What is the remainder when $x^{13} + 7x^5 - 3x^2 + 4$ is divided by $x + 1$?

22. Given that $-2$ is a root of the equation $2x^3 - x^2 - 7x + 6 = 0$, find the other two roots.

23. Find the solution set of the inequaltiy $2x^2 < x + 15$ and graph this solution set on the real number line.

24. Find the solution set of the euqation

$$|x + 1|^2 + |x + 1| = 6$$

25. Graph the solution set of $|x| + 2|y| < 5$ on the number plane.

Answers (Grade 11, Chapter 4)

1. $\{-3, 2\}$
2. (a), (c)
3. $\frac{14}{11}$
4. 49, 51
5. $\pm 3$
6. 0
7. $\frac{-a \pm \sqrt{a^2 - 4bc}}{2c}$
8. 4
9. Parabola opening downward;
   x - intercepts: $-1, \frac{3}{2}$;
   y - intercepts: 0
10. $k < \frac{1}{3}$
11. $5x^2 - 13x - 6 = 0$
12. $10x^2 + 11x - 6 = 0$
13. $-\frac{5}{3}, -\frac{7}{3}$
14. $\frac{2}{3}$
15. $\{\frac{1}{2}\}$
16. 82
17. $\{-1,1,2,4\}$
18. $128(3 + 2\sqrt{2})$

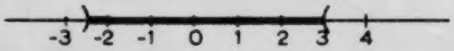
19. $\{-3, -2, -1, 0, 1, 2, 3, 4\}$

20. (a), (b), (c).

21. $-7$

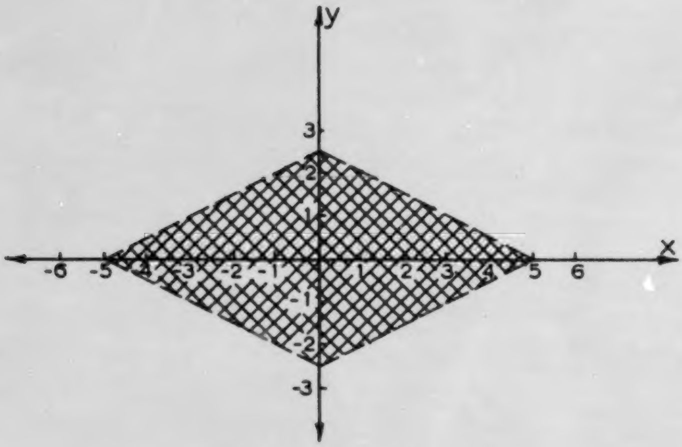
22. $1, \frac{3}{2}$

23. $\{x: -\frac{5}{2} < x < 3\}$



24. $\{-3, 1\}$

25.



# Chapter 5

## Teachers Manual

## The Complex Number System

### Commentary on the Text.

The complex number system is one of the supreme achievements of the human intellect. Compelling reasons for extending the real number system are easy to find. In the context of the real number system the theory of quadratic equations is most unsatisfactory, for some quadratic equations with real coefficients have real solutions, while others have no real solutions. The desire to remedy this situation is surely reasonable and modest. What is remarkable is the fact that this modest aim, once attained, yields a system so rich that no further extensions are necessary to capture the roots of any algebraic equation of whatever degree. However, the solution of algebraic equations is only one of the achievements of the complex number system. It is surely lamentable that we are unable, at this level of the students' development, to indicate the profusion of important and beautiful results to be found in the theory of functions of a complex variable. We can only state--with all the enthusiasm we can muster--that this field of mathematics (and others closely related to it) is probably the most intensively cultivated at the present time, and that its applications in the sciences and engineering seem to grow daily.

## 5-1 Comments

In Section 5-1 we review the shortcomings of the real number system with regard to the problem of solving quadratic equations, and we reveal our intention to attempt a remedy by extending the real number system. The exercises may be used for an oral review of number systems.

## 5-2 Comments

In Section 5-2 we make the extension. This section falls into two parts, corresponding to the two tasks we face each time we extend a number system. These tasks are (1) to set down the properties we require of our new system (P-1, P-2, P-3, P-4), and (2) to deduce the calculation rules of the new system (Theorems 5-1, 5-2, 5-3) from the fundamental properties.

The particular list of fundamental properties which we have adopted is by no means the only one we could have chosen. We do, indeed, assume much more than we need to, but a more economical selection of assumptions would require a greater attention to subtleties than seems appropriate to an introductory study at this level. (Students whose curiosity is aroused in this connection may be referred to Chapter 15 where considerable care is devoted to questions of this sort.) Some of the assumptions we made, but need not have made, are (1) that there is only one additive identity, (2) that there is only one multiplicative identity, (3) that these identity elements coincide with the corresponding identity elements of R, (4) that additive inverses are unique. Such questions of uniqueness are rather subtle to beginning students. They are, never the less, important and fundamental. We face the issue of uniqueness first in proving that $0 \cdot z = 0$. We face it again in the discussion of subtraction. (We face it once more in Section 5-4 when we discuss division.) It is absolutely necessary that the student understand the difference between saying that a first degree equation, for example, has at most one solution and saying that is has at least one solution. In the old-fashioned terminology, "solving for x" is the process of showing there is at most one solution, while "checking" is the process of showing that there is at least one solution.

The exercises illustrate the calculation rules, offering the students an opportunity to develop facility in the manipulation of sums, differences, and products of complex numbers. Exercise 8 asks the student to show that the equation $z_1 + z = z_2$, $z_1$, $z_2$ given, has at most one solution. The teacher may find it desirable to work this exercise in class to illustrate the question of the uniqueness of the solution of a simple equation. Suppose, indeed, that $z_3$ and $z_4$ each satisfy the equation:

$$z_1 + z_3 = z_2 \quad \text{and} \quad z_1 + z_4 = z_2.$$

We are to show that $z_3 = z_4$, i.e., we cannot have two different solutions. Add $-z_1$, the additive inverse of $z_1$, to each equation:

$$(-z_1) + z_1 + z_3 = z_2 + (-z_1); \quad (-z_1) + z_1 + z_4 = z_2 + (-z_1).$$

So $$z_3 = z_2 + (-z_1) \text{ and } \qquad z_4 = z_2 + (-z_1),$$

and $z_3$, $z_4$ must be equal. It must be emphasized that this argument does not show that $z_1 + z = z_2$ has any solution; it shows only that it has not got more than one solution. The calculation in the text shows that it has at least one--it does this by showing that $z_2 + (-z_1)$ actually satisfies the equation.

## 5-3 Comments

Section 5-3 is devoted to proving Theorem 5-4 and to defining some important terms. Theorem 5-4 asserts that each complex number z may be written in the form a + bi (a and b real) in only one way. (P-4 asserts that z may be written in this form in at least one way.) This theorem justifies the definite article in the expression "the standard form" used to describe this way of writing z. (One advantage of Theorem 5-4 is that it shows us we can have only one answer for exercises like those in Section 5-2 where the student is asked to express certain complex numbers in what we now call "standard form".) The double-barrelled way in which Theorem 5-4 is stated gives the teacher an opportunity to refresh the students' minds on the distinction between "if" and "only if", a distinction which seems to be impossible to overemphasize. The proof itself is broken into two parts. In the first part we prove a special case of the theorem; in part two our proof of the theorem itself hinges on our previous proof of the special case. (This type of proof--which is perfectly valid--raises an interesting logical point which may occur to some of the brighter students. Notice that the special case implies the theorem itself. Of course, the theorem implies its special case. This means that the theorem and its special case are logically equivalent (i.e., each imples the other). Thus, though they look very different, the theorem apparently asserting much more than the special case, the proof shows that the theorem and its special case have the same logical force. We do not mean to suggest that the teacher volunteer this observation; we only disucss it here to prepare the teacher in the event some student should notice it and bring it up.)

A word (or two) about the terms defined in Section 5-3. "Standard form" should cause no trouble; though one must emphasize that the $a$ and $b$ appearing in the standard form are *real numbers*. (Throughout the rest of the chapter we sometimes say "$a + bi$, in standard form" and sometimes "$a + bi$, where $a$ and $b$ are real numbers", these expressions have identical meanings.) "Real part" is straight-forward and causes no trouble. Mathematicians have used the expression "imaginary part" for many years as we define it in the text, so that the imaginary part of a complex number is a *real* number. This terminology may be unfortunate, but it is standard. Writers of many elementary books have departed from the mathematicians' usage, saying that $bi$ is the "imaginary part" of $a + bi$. Many students will be reading other books and will notice that they are not all in agreement. (This experience is a valuable part of anyone's education.) A student who goes on in mathematics has to learn sooner or later that $b$ is called the imaginary part of $a + bi$ in advanced work. Since it seems a shame to teach him something he must later unlearn, we stick to the mathematicians' standard terminology: the imaginary part of a complex number is a real number.

## 5-4 Comments

In Section 5-4 we attack the problem of division. As stated in the text, our fundamental properties in Section 5-2 are silent on the questions of quotients and multiplicative inverses of elements of C, whether some exist in C and if they do whether they are unique. A case could be made for simply postulating all we want to know about inverses and quotients, but we do not believe that a very good case can be made for this position. If we simply "postulate" everything, leaving nothing to derive as theorems, we no longer have a mathematical postulate system, but a list of dogmatic pronouncements. As we indicated above--in the comments on Section 5-2 --we could have assumed less and proved more. We take an intermediate position, supposing all we can about sums, products and differences. This enables us to get started on the manipulation with complex numbers. By supposing nothing about inverses and quotients of elements of C we leave something we can do to justify calling our development postulational.

We begin recalling the solution of the equation $ax = b$, $a \neq 0$, in the real number system. We use the same method of solution in the case of the complex number system. It is worth noting that at a crucial step in our argument (from $(a^2 + b^2)x = a$ to $x = a/(a^2 + b^2)$, and similarly with y) we use the fact that the real number system contains a multiplicative inverse for each non-zero real number. The step is justified by the fact that we have already equated real and imaginary parts (using Theorem 5-4) and are therefore working in the real number system R. As long as we are working in R we have at our disposal all of the results in Chapter 1, in particular we know that each non-zero element of R has a multiplicative inverse in R. In this way, we use a property of R to prove that C has the corresponding property.

We close the section by announcing three results, familiar from the study of R, which can now be carried over to C. Their proofs--they are formally the same as in R--are left to Exercises 6 and 7.

Exercises 1, ···, 5 offer practice in applying the general formulas in the text for multiplicative inverses and quotients. Exercise *8 concerns a subtle point in connection with Theorem 5-5; presumably it would be of interest only to some of the better students.

General comments on Sections 5-5, 5-6, 5-7. These three sections are more or less independent of one another (except as noted below) and may be taken in any order appealing to the teacher or depending on the interests of the class. There is a reference to conjugates in Section 5-6, following Theorem 5-7, but no knowledge of conjugates is needed to understand the rest of Section 5-6. Absolute values are not mentioned again until Section 5-7, Graphical Representation. Some teachers may therefore wish to postpone Absolute Values until a geometric interpretation is available. (In the case of a group more interested in geometry than in algebra this course would be advisable as it provides a geometric motivation for this essentially algebraic notion.)

## 5-5 Comments

In Section 5-5 we define the complex conjugate and the absolute value of a complex number, and discuss some of their properties. Since conjugates are referred to only in passing in later sections of this chapter and absolute values do not appear again until Section 5-7, in a class pressed for time it will be possible to treat Section 5-5 very lightly at first, although careful attention to these matters is suggested before studying Chapter 14, Polar Form of Complex Numbers.

A number of very interesting questions come up in connection with complex conjugates and absolute value and some teachers may expect a good deal of student interest in them. The notion of a conjugate is something pecular to the system C, since in R each number is its own conjugate. Indeed those elements of C not in R are precisely those which are not equal to their conjugates. The definition of absolute value in C should appeal to many students because it extends a notion, familiar from the study of R, and it does it in a way which seems quite striking when first encountered. The fact that the important properties of absolute value (absolute value of a product, absolute value of a sum) carry over completely may almost seem too good to be true. It is curious that algebraically the product formula, $|z_1z_2| = |z_1|\cdot|z_2|$, is quite easy to prove while the inequality for the sum, $|z_1 + z_2| \leq |z_1| + |z_2|$, is really rather subtle. (Geometrically, it is just the other way round, the sum is handled easily while the product is more elusive.) Using the relation connecting a complex number, its conjugate and its absolute value, we give a simple algebraic proof of the product formula in the text. In some of the starred exercises we lead any students who are both interested and persistent through algebraic arguments for the inequality for sums. (Very likely the manipulations involved will not appeal to many students.) Studying absolute values provides an excellent opportunity to do a little reviewing of inequalities, and for this reason alone many teachers will find their time well spent here.

## 5-6 Comments

Section 5-6 treats the theory of quadratic equations with real coefficients, extending the "quadratic formula" to cover the case of a negative discriminant. Since this formula involves the expression $\sqrt{b^2 - 4ac}$ and we are interested in the case $b^2 - 4ac < 0$, we are obliged to precede our discussion of the formula by a definition of $\sqrt{r}$ for $r$ real and negative. Hence we begin with the examples $z^2 + 1 = 0$, $z^2 + r = 0$ leading up to the extended definition of $\sqrt{r}$ (Defintion 5-3). With the definition of $\sqrt{r}$ available, we summarize our results on the special quadratics (having no first degree term) in Theorem 5-6, a result we need in the proof of Theorem 5-7. Theorem 5-7 is proved by the usual process of completing the square, then using Theorem 5-6 we solve

$$(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}.$$

Since $\sqrt{4a^2} = 2|a|$, the square roots of the right member are

$$\frac{\sqrt{b^2 - 4ac}}{2|a|}, \qquad \frac{-\sqrt{b^2 - 4ac}}{2|a|};$$

one of which is $\frac{\sqrt{b^2 - 4ac}}{2a}$, the other $\frac{-\sqrt{b^2 - 4ac}}{2a}$ (though which is which depends on whether $a > 0$ or $a < 0$). Theorem 5-7 solves a problem we set out to solve: finding the roots of the general quadratic equation with real coefficients. We find that a quadratic with real coefficients is one of only three types: (1) It has one root--which is real--if its discriminant is zero, (2) It has two (different) real roots if its discriminant is positive, (3) It has two (different) non-real complex roots--which are conjugates of each other--if its discriminant is negative.

> [A word to the teacher on quadratic equations with complex coefficients.
>
> We state in the text that each quadratic equation with complex coefficients has all its roots in C. Suppose
>
> (*) $$\alpha z^2 + \beta z + \gamma = 0$$
>
> is such an equation--$\alpha, \beta, \gamma$ denoting given elements of C, $\alpha \neq 0$. Let us proceed as in the case of real coefficients, completing the square:
>
> (**) $$(z + \frac{\beta}{2\alpha})^2 = \frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}.$$
>
> We now find there are two cases, according as the complex number

$$\frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}$$

is (1) real, (2) not real. If the right-hand member of (**) is real (Case 1.), we can proceed exactly as in the text where the coefficients are all real:

(***) $$z + \frac{\beta}{2\alpha} = \sqrt{\frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}} \quad \text{or} \quad z + \frac{\beta}{2\alpha} = -\sqrt{\frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}}\ .$$

for the radicand is a real number. Now $\alpha$ itself is either real or not; if it is not we can convert to an equivalent equation (multiplying each coefficient of (*) by $\bar{\alpha}$) in which the coefficient of $z^2$ is real. Let us assume, then, that the $\alpha$ in (***) is real. We can now write

$$z + \frac{\beta}{2\alpha} = \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad \text{or} \quad z + \frac{\beta}{2\alpha} = -\frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\ .$$

(if $\alpha < 0$ we have interchanged the order of writing this pair of equations; if $\alpha > 0$ we have not.) We therefore see that the expressions given in Theorem 5-7 are valid even if the coefficients $\alpha, \beta, \gamma$ are complex provided $\alpha$ and $\beta^2 - 4\alpha\gamma$ are real numbers. (Exercise 14, Section 5-6, is an example of this kind: $\alpha = 1$, $\beta^2 - 4\alpha\gamma = (-i)^2 - 4(1)(2) = -1 - 8 = -9$. We give other examples below.) Note that the roots of (*) being

$$\frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \quad \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$

where $\alpha$ and $\beta^2 - 4\alpha\gamma$ are real, are not complex conjugates of each other if $\beta$ is not real. (In Exercise 14, the roots are

$$\frac{i + \sqrt{-9}}{2} = \frac{i + 3i}{2} = 2i$$

and $$\frac{i - \sqrt{-9}}{2} = \frac{i - 3i}{2} = - i)$$

The "conjugacy" of the roots, hinging as it does on the alternate signs of the radical, requires $\alpha$ and $\beta$ (which are outside the radical) to be real. (As a matter of fact if $\alpha, \beta$ and $\beta^2 - 4\alpha\gamma$ are all real, so is $\gamma$, for $\beta^2 - 4\alpha\gamma = r$ gives $\gamma = \frac{\beta^2 - r}{4\alpha}$.)

Summarizing: If $\alpha$ and $\beta^2 - 4\alpha\gamma$ are both real--though $\beta$ or $\gamma$, or both, be complex--the ordinary formulas give the roots of $\alpha z^2 + \beta z + \gamma = 0$. But the two roots are conjugates in this case, if and only if $\beta$ and $\gamma$, as well as $\alpha$ and $\beta^2 - 4\alpha\gamma$, are both real and $\beta^2 - 4\alpha\gamma < 0$.

For case (2), $\frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}$ non-real, we have (as yet) no way to get from equation (**) to anything like equations (***). We must postpone this case until Chapter 14, Polar Form of Complex Numbers.

Examples of quadratic equations with complex coefficients falling in Case (1), $\frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}$ real, may be obtained by replacing a, b, c, $\beta$, $\gamma$ below by numerical values satisfying the conditions listed:

(I) $az^2 + (bi)z + c = 0$,

where a, b, c are real.

(II) $z^2 + \beta z + \gamma = 0$,

where $\beta$, $\gamma$ are complex and $\gamma = r + \frac{1}{4}\beta^2$, r being a real number.

There is another way to attack the equation

$$\alpha z^2 + \beta z + \gamma = 0, \ \alpha \neq 0, \ \alpha, \beta, \gamma \text{ in } C.$$

We may convert the problem into a problem involving only real coefficients and real "unknowns" if we write

$$\left.\begin{aligned} \alpha &= a_0 + a_1 i \\ \beta &= b_0 + b_1 i \\ \gamma &= c_0 + c_1 i \\ z &= x + yi \end{aligned}\right\} \quad \text{where } a_0, a_1, b_0, b_1, c_0, c_1, x, \text{ and } y \text{ are real}$$

and separate the real and imaginary parts:

$$\begin{aligned} \alpha z^2 + \beta z + \gamma &= (a_0 + a_1 i)(x + yi)^2 + (b_0 + b_1 i)(x + yi) + (c_0 + c_1 i) \\ &= (a_0 + a_1 i)(x^2 - y^2 + 2xyi) + (b_0 x - b_1 y) \\ &\qquad + (b_1 x + b_0 y)i + (c_0 + c_1 i) \\ &= a_0(x^2 - y^2) - 2a_1 xy + b_0 x - b_1 y + c_0 \\ &\qquad + (a_1(x^2 - y^2) + 2a_0 xy + b_1 x + b_0 y + c_1)i. \end{aligned}$$

So $\alpha z^2 + \beta z + \gamma = 0$ if and only if

$$\begin{cases} a_0 x^2 - 2a_1 xy - a_0 y^2 + b_0 x - b_1 y + c_0 = 0 \\ a_1 x^2 + 2a_0 xy - a_1 y^2 + b_1 x + b_0 y + c_1 = 0. \end{cases}$$

This process yields a pair of simultaneous second degree equations in the two real unknowns $x$ and $y$. Except in special cases, however, the problem of solving this pair of simultaneous equations is quite tedious. (Some of the simpler special cases appear in Exercises 4, Section 5-3; and 17, 18, Section 5-6.) Using methods of analytic geometry, it is possible to show that each of the second degree equations of the system represents a hyperbola (or a pair of intersecting lines) and that these graphs are so situated that they have exactly two points of intersection unless $\beta^2 - 4\alpha\gamma = 0$, when there is only one point of intersection. In this way one may show that each quadratic equation with complex coefficients does have one or two complex roots. Except for the simplest cases, however, it is much easier to find the roots using the methods discussed in Chapter 14.]

The exercises for Section 5-6 provide practice in using Theorem 5-7 and in calculating with the square-root symbol. Exercises 1,2, *3 concern square roots, while Exercises 4, ···, 15 are quadratics to be solved. Exercise 16 involves a special cubic which is easily reduced to a quadratic. (It provides an illustration of the Fundamental Theorem of Algebra--Cf. Section 5-8--asserting that each polynomial equation with complex coefficients has $n$ roots if the polynomial has degree $n$.) Exercises 17 and 18 were discussed in the bracketed remarks above. Exercise 19 presents the famous relations between the roots of a quadratic equation and its coefficients ("Newton's Relations"). Warning to the teacher: As stated in the text, this exercise is a booby trap. We intentionally neglect to restrict $a$, $b$, $c$ to be real numbers. (We do this so we can apply the more general result in Exercises 20 and 21.) A teacher wishing to spare his students (as well as himself!) some mental anguish may simply ask the students to assume $a$, $b$, $c$ real, but announce that this assumption is unnecessary. In this way the students can proceed with Exercises 20 and 21 using the less restricted version of Exercise 19. Exercise 19--as it stands in the text--might be used as a starred exercise for some of the more eager students.

## 5-7 Comments

In Section 5-7 we study the graphical representation of of complex numbers by means of so-called Argand diagrams. (This method of plotting complex numbers $z = x + yi$, $x$, $y$ real, in the xy-plane was actually done first by a man named Wessell (1797). Argand did the same thing--independently of Wessell--a few years later (1806). But Wessell's work remained unknown for a century, and mathematicians had become accustomed to the designation "Argand diagram" by the time they learned better (1897). Thus we have an example of one of those historical mistakes that everyone passes along, knowing better.) The idea involved in the Wessell-Argand graphical representation is natural enough to anyone familiar with coordinate geometry in two dimensions. We set up a one-to-one correspondence between $z = x + yi$, in standard form, and the point $(x,y)$ in the xy-plane. The formula for the sum of two complex numbers (Theorem 5-1, Section 5-2) gives the "parallelogram law" for adding two complex numbers geometrically. (This law will be met again, in another context, in Chapter 13, vectors.) A geometrical representation of products--not mentioned in Chapter 5--will be found in Chapter 14, as a knowledge of trigonometry is required.

The first eighteen exercises involve plotting a number of complex numbers in Argand diagrams. (Some teachers, wishing to emphasize geometrical constructions, may be disposed to change the instructions for some of Exercises 3, ··· 18, asking the students to find the sums and differences involved geometrically, checking their constructions algebraically.) Exercise 19 asks the student to justify an assertion made in the text without proof. Exercise 20 calls for a geometrical proof of the "triangle inequality" 5-51. (Cf. Exercises *12, *13, Section 5-5.) The final three exercises introduce the student to a little of the geometry of the Argand diagram.

## 5-8 Comments

Section 5-8 surveys some of the things we have done in Chapter 5 and indicates many things we have not done. Here we announce the Fundamental Theorem of Algebra, without proof, and argue (rather tersely) how it shows that C is "algebraically closed": All the roots of every polynomial equation, with coefficients in C, are elements of C.

Following Section 5-8 there is a set of Miscellaneous Exercises which may be used to review the work of this chapter.

## 5-9 Comments

Section *5-9 (Appendix, page 33) outlines Gauss's construction of the complex number system. As a source of historical information we suggest The Development of Mathematics, by E. T. Bell (McGraw-Hill, 1945, Second Edition): Wessell and Argand, p. 177; Gauss, p. 179; Cauchy, p. 194.

## Exercises 5-1 Answers

1. No, yes, yes
2. Natural numbers were extended to a system (the integers) including 0 and negative integers.
3. No
4. Rational number system.
5. No (If $x$ is real, then $x^2 \geq 0$, so $x^2 + 3 \geq 0 + 3 = 3 > 0$.)

## Exercises 5-2 Answers

1. (a) $0 + 0i$
   (b) $1 + 0i$
   (c) $-1 + 0i$
   (d) $0 + (1)\ i$
   (e) $-1 + 0i$
   (f) $0 + (-1)i$
   (g) $1 + (-1)i$
   (h) $3 + 0i$
   (i) $0 + 2i$

2. (a) $4 + 9i$
   (b) $4 + 0i$
   (c) $3 + 6i$
   (d) $(4 + \pi) + \pi i$
   (e) $(1 + \sqrt{2}) + 5i$
   (f) $-1 + 7i$
   (g) $8 + (1)i$
   (h) $0 + 7i$
   (i) $3 + 7i$

3. (a) $(2 - 5i) + (r + 5i) = 2 + r$, $r$ any real number .
   (b) If $x, y$ are assumed to be real,
   $(x - yi) + (r + yi) = x + r$, $r$ any real number.
   If $x, y$ are not assumed to be real, let
   $x = a + bi$, $y = c + di$ ($a, b, c, d$ real), then
   $x - y = (a - c) + (b - d)i$ and
   $(x - y) - [r + (b - d)i] = a - c + r$, $r$ any real number

(c) $(\sqrt{2} - \sqrt{3}\,i) + (r + \sqrt{3}\,i) = \sqrt{2} + r$, $r$ any real number

(d) $-5i + (r + 5i) = r$, $r$ any real number.

4. (a) $-13 + 26\,i$

(b) $24 + (-10)i$

(c) $5 + 5i$

(d) $-5 + 3i$

(e) $2 + (2\sqrt{2})i$

(f) $(8 - \sqrt{6}) + (\sqrt{2} + 8\sqrt{3})i$

(g) $-7 + 24\,i$

(h) $2 + 0i$

5. (a) $-3 = -3 + 0i$

(b) $-i = 0 + (-1)i$

(c) $-1 = -1 + 0i$

(d) $-2 - 3i = -2 + (-3)i$

(e) $-5 - 3i = -5 + (-3)i$

(f) $-4 + 3i$

(g) $-a + bi$ (This is in the required form if $a$, $b$ real)

(h) $-x - yi = -x + (-y)i$, if $x$, $y$ real

6. (a) $5 + 8i$

(b) $-2 + 2i$

(c) $0 + 10i$

(d) $-1 + 0i$

(e) $(\sqrt{3} - 2) + (1 - \sqrt{2})i$

(f) $1 + i$

(g) $\pi + (-\pi)i$

(h) $0 + 6i$

(i) $1 + (-3)i$

7. First $i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

$i^8 = 1$

etc.

In general: for $n = 4m$ ($m$ a natural number), $i^n = i^{4m} = (i^4)^m = 1^m = 1$; for $n = 4m + 1 (m \in N)$, $i^n = i^{4m+1} = i^{4m}i = i$; for $n = 4m + 2$ $(m \in N)$, $i^n = i^{4m+2} = i^{4m}(-1) = -1$; for $n = 4m + 3$ $(m \in N)$, $i^n = i^{4m+3} = i^{4m}i^3 = -i$. These possibilities are all there are, for if $n$ is a natural number and we divide it by 4 the only non-negative remainders less than 4 which we can get are 0, 1, 2, 3.

8. (See 5-2 Comments) The proof is as follows: If $z$ is a complex number satisfying $z_1 + z = z_2$, then $(-z_1) + z_1 + z = z_2 + (-z_1)$ so $z = z_2 + (-z_1)$. Thus no complex number other than $z_2 + (-z_1)$ can satisfy the equation. (It is shown in the text that this number really does satisfy the equation.)

Exercises 5-3 Answers

| | | Real Part | Imaginary Part |
|---|---|---|---|
| 1. | (a) | 0 | 2 |
| | (b) | 0 | 0 |
| | (c) | 0 | 1 |
| | (d) | 5 | -1 |
| | (e) | $2x_0$ if $x_0$ real part of $x$ | $3 + 2x_1$, if $x_1$ imaginary part of $x$ |
| | (f) | $a_0$, if $a_0$ real part of $a$ | $-2 + a_1$, if $a_1$ imaginary part of $a$ |
| | (g) | 1 | $-2\sqrt{2}$ |
| | (h) | -2 | $-2\sqrt{3}$ |
| | (i) | -3 | 1 |
| | (j) | 2 | 0 |
| | (k) | 0 | 3 |
| | (l) | 1 | 2 |

2 (a) -3

(b) - 5

(c) 0

(d) -5

(Only one way in each case)

3. (a) $x = 3$, $y = -6$

(b) $x = 3$, $y = 0$

(c) $x = 0$, $y = -4$

(d) $x = \frac{1}{2}$, $y = -3$

(e) $x = -\frac{4}{3}$, $y = 2$

4. $(x + iy)^2 = -7 + 24i$

$(x^2 - y^2) + 2xyi = -7 + 24i$

$x^2 - y^2 = -7$

$2xy = 24$

$y = \frac{12}{x}$ (Note that neither x or y may be 0.)

$x^2 - (\frac{12}{x})^2 = -7$

$x^4 - 144 = -7x^2$

$x^4 + 7x^2 - 144 = 0$

$(x^2 + 16)(x^2 - 9) = 0$

$x^2 = 9$

$x_1 = +3$, $y_1 = 4$

$x_2 = -3$, $y_2 = -4$

5. (a) $8 + 3i$

(b) $-2 + 0i$

(c) $6 + 12i$

(d) $4 + 8i$

(e) $11 + (-16)i$

(f) $10 + (-11)i$

(g) $18 + 14i$

(h) $(a^2 + 2ab + b^2 + c^2) + 0i$

6. Suppose $a = c$ and $b = d$. Then $bi = di$ and $a + bi = c + di$. (This argument is good whether a, b, c, d are real or complex numbers.) If $a = a_0 + a_1i$, $b = b_0 + b_1i$, $c = c_0 + c_1i$, $d = d_0 + d_1i$ ($a_0, a_1, \cdots, d_1$ real), then (using Theorem 5-4) $a + bi = c + di$, meaning

$(a_0 + a_1 i) + (b_0 + b_1 i)i = (c_0 + c_1 i) + (d_0 + d_1 i)i$ or $(a_0 - b_1) + (a_1 + b_0)i = (c_0 - d_1) + (c_1 + d_0)i$, holds if and only if

$$a_0 - b_1 = c_0 - d_1 \quad \text{and } a_1 + b_0 = c_1 + d_0.$$

Thus examples must satisfy these conditions. (A simple example is $a_0 = 1$, $d_1 = -1$, the others 0:

$$(1 + 0i) + (0 + 0i)i = (0 + 0i) + (0 - i)i.)$$

*7. Let $a = a_0 + a_1 i$, $b = b_0 + b_1 i$; $a_0, a_1, b_0, b_1$ real. Then $a = 0$ means $a_0 = a_1 = 0$ and $b = 0$ means $b_0 = b_1 = 0$ (by the "special case" of Theorem 5-4). Consider the statements

$$a + bi = (a_0 - b_1) + (a_1 + b_0)i = 0,$$

and

$$a - bi = (a_0 + b_1) + (a_1 - b_0)i = 0.$$

The first holds, if and only if $a_0 = b_1$ and $a_1 = -b_0$; the second, if and only if $a_0 = -b_1$ and $a_1 = b_0$. Thus the two together are equivalent to

$$a_0 = b_1 = -b_1 \quad \text{and } a_1 = -b_0 = b_0.$$

But $b_1 = -b_1$ means $b_1 = 0$; since $a_0 = b_1$, $a_0 = 0$ too. Similarly for $a_1, b_0$. But $a_0 = a_1 = b_0 = b_1 = 0$ if and only if $a = 0$ and $b = 0$, by the "special case" of Theorem 5-4 again. As to replacing the underlined "and" by "or", note that $a + bi = 0$, $a - bi = 0$ are equivalent, respectively to $a = -bi$, $a = bi$. That these need not hold simultaneously ($a$, $b$ non-real) may be seen by taking any non-zero complex number for $b$. Multiplying it in turn by $i$ and $-i$ will give two different values for "a". For example $b = 1$, $a = -i$; and here $a + bi = -i + (1)i = 0$, while $a \neq 0$, $b \neq 0$, $a - bi = -i - i = -2i \neq 0$.

For $b = 1$, $a = i$, we have $a - bi = i - i = 0$, but $a \neq 0$, $b \neq 0$, $a + bi = 1 + i \neq 0$.

Exercises 5-4 Answers

1. (a) $1 = 1 + 0i$

   (b) $\frac{1}{5} = \frac{1}{5} + 0i$

   (c) $-i = 1 + (-1)i$

   (d) $i = 1 + (1)i$

   (e) $\frac{1}{2} - \frac{1}{2} = \frac{1}{2} + (-\frac{1}{2})i$

   (f) $\frac{2}{13} - \frac{3}{13}i = \frac{2}{13} + (\frac{-3}{13})i$

   (g) There is no multiplicative inverse since $1 + i^2 = 0$

   (h) $\frac{4}{25} + \frac{3}{25}i$

2. This amounts to solving the equation

$$a = \frac{1}{a}$$

$$a^2 = 1$$

$$a = +1 \text{ and } a = -1$$

3. This amounts to solving the equation

$$a = -\frac{1}{a}$$

$$a^2 = -1$$

$$a = i \quad \text{and } a = -i$$

4. (a) $\frac{2}{5} - \frac{1}{5}i = \frac{2}{5} + (\frac{-1}{5})i$

   (b) $-\frac{3i}{2} = 0 + (-\frac{3}{2})i$

   (c) $1 - i = 1 + (-1)i$

   (d) $-\frac{9}{5} - \frac{12}{5}i = -\frac{9}{5} + (-\frac{12}{5})i$

   (e) $-\frac{3}{5}i = 0 + (-\frac{3}{5})i$

   (f) $\frac{-5 - 2i}{29} = -\frac{5}{29} + (\frac{-2}{29})i$

   (g) $\frac{5 - 13i}{2} = \frac{5}{2} + (-\frac{13}{2})i$

   (h) $\frac{1 + 2\sqrt{6}\,i}{2} = \frac{1}{5} + (\frac{2\sqrt{6}}{5})i$

(i) $\frac{-4}{5} + \frac{2}{5}i$

(j) $\frac{2(a^2 - b^2)}{4a^2 + b^2} + \frac{5ab}{4a^2 + b^2}\, i$

(k) $\frac{3x^2 - 2y^2}{x^2 + y^2} + \frac{5xy}{x^2 + y^2}\, i$

(l) $\frac{m^2 - n^2}{m^2 + n^2} + \frac{2mn}{m^2 + n^2}\, i$

(m) $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}\, i$

(n) $\frac{x^2 - 4y^2}{x^2 + 4y^2} + \frac{4xy}{x^2 + 4y^2}\, i$

5. $z = a + bi$ (Note that $a^2 + b^2 \neq 0$.)

$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\, i$

Real part of $\frac{1}{z} = \frac{a}{a^2 + b^2} = \frac{1}{2}$

$$2a = a^2 + b^2$$

(a) If $b = 0$, then $a = 2$ (since a cannot be 0 too)

$$z = a + bi = 2$$

(b) If $b = \frac{1}{2}$, then $2a = a^2 + \frac{1}{4}$

$$4a^2 - 8a + 1 = 0$$

$$a = 1 + \frac{\sqrt{3}}{2}\text{, or}$$

$$a = 1 - \frac{\sqrt{3}}{2}.$$

$Z_1 : 1 + \frac{\sqrt{3}}{2} + \frac{i}{2}$ and $Z_2 : 1 - \frac{\sqrt{3}}{2} + \frac{i}{2}$

(c) If $b = 1$, then $2a = a^2 + 1$

$$a^2 - 2a + 1 = 0$$

$$a = 1$$

$Z : 1 + i$

6. "If" part follows immediately from formula 5-2a. "Only if": Suppose $z_1z_2 = 0$. If $z_1 \neq 0$, then $\frac{1}{z_1}$ exists in C. Multiplying by $\frac{1}{z_1}$, we get $z_2 = 0\cdot\frac{1}{z_1} = 0$, by formula 5-2a again. Thus if $z_1z_2 = 0$ but $z_1 \neq 0$, we have $z_2 = 0$. Similarly, if $z_1z_2 = 0$ but $z_2 \neq 0$, we get $z_1 = 0$. So $z_1z_2 = 0$ implies $z_1 = 0$ or $z_2 = 0$. (If neither $\frac{1}{z_1}$ nor $\frac{1}{z_2}$ is in C, then $z_1 = 0$ and $z_2 = 0$, by Theorem 5-5.)

7. Formula 5-4g. Let $w_0$ be the unique solution of $z_2w = z_1$; then $w_0 = \frac{z_1}{z_2}$ and $z_2w_0 = z_1$. Similarly let $w_1$ be the unique solution of $z_4w = z_3$; then $w_1 = \frac{z_3}{z_4}$ and $z_4w_1 = z_3$. Also let $w_2$ be the unique solution of $(z_2z_4)w = z_1z_3$; then $w_2 = \frac{z_1z_3}{z_2z_4}$. We must show that $w_0w_1 = w_2$. From $z_2w_0 = z_1$ and $z_4w_1 = z_3$ we get $(z_2w_0)(z_4w_1) = z_1z_3$ or $(z_2z_4)(w_0w_1) = z_1z_3$. Thus $w_0w_1$ satisfies $(z_2z_4)w = z_1z_3$. But $w_2$ is the only solution of this equation. Hence $w_0w_1 = w_2$.

   Formula 5-4h. With $w_0 = \frac{z_1}{z_2}$ and $w_1 = \frac{z_3}{z_4}$ as above, let $w_3$ be the unique solution of $(z_2z_4)w = z_1z_4 + z_2z_3$. To show $w_0 + w_1 = w_3$. From $z_2w_0 = z_1$ and $z_4w_1 = z_3$ we get $z_4(z_2w_0) = z_1z_4$ and $z_2(z_4w_1) = z_2z_3$; so, adding, $z_2z_4(w_0 + w_1) = z_1z_4 + z_2z_3$. Thus $w_0 + w_1$ satisfies the equation whose only root is $w_3$. Hence $w_0 + w_1 = w_3$.

*8. Whether or not a and b are real numbers, provided that $a^2 + b^2 \neq 0$, we can multiply the factors $a + bi$, $\frac{a - bi}{a^2 + b^2}$ and get $(a + bi)\,\frac{a - bi}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} + \frac{ab - ab}{a^2 + b^2}\,i = 1$

(for there is nothing in the proof of Theorem 5-2 which requires a and b to be real). Thus $\frac{a - bi}{a^2 + b^2}$ is an inverse of $a + bi$, if $a^2 + b^2 \neq 0$. But we know already that no complex number can have more than one inverse, for if it did Theorem 5-5 (as stated in the text) would be false.

Exercises 5-5 Answers

1. (a) $2 - 3i = 2 + (-3)i$

(b) $-3 - 2i = -3 + (-2)i$

(c) $1 + i = 1 + (1)i$

(d) $-5 + 0i$

(e) $2i = 0 + 2i$

(f) $0 + 0i$

(g) $\pi + 0i$

(h) $4 - i = 4 + (-1)i$

(i) $-3 - 4i = -3 + (-4)i$

2. (a) 5

(b) 13

(c) 1

(d) 3

(e) $\sqrt{\frac{13}{5}}$

(f) $\frac{2}{\sqrt{29}}$

3. (a) $4 + 7i$

(b) $3 + 4i$

(c) $\frac{3}{2} + (-\frac{1}{2})i$

(d) $10 + (-4)i$

(e) $\frac{1}{10} + \frac{3}{10}i$

(f) $\frac{1}{2} + (-\frac{3}{4})i$

(g) $-\frac{1}{13} + (-\frac{5}{13})i$

(h) $\frac{1}{2} + \frac{1}{2}i$

(i) $\frac{2a^2 + 3b^2}{4a^2 + 9b^2} + \frac{-ab}{4a^2 + 9b^2}i$

(j) $\frac{2x - y^2}{4x^2 + y^2} + \frac{3xy}{4x^2 + y^2}i$

4. Let $z = a + bi$, in standard form. Then $z$ is a pure imaginary number if and only if $a = 0$. On the other hand $\bar{z} = a - bi$, thus $z = -\bar{z}$ if and only if

$$(*) \qquad a + bi = -a + bi.$$

By theorem 5-4, (*) holds if and only if

$$a = -a \text{ and } b = b,$$

and these **hold** if and only if $a = 0$.

5. Definition 5-1 states that $a - bi$ is the conjugate of $a + bi$ provided $a$ and $b$ are real. Thus $a - bi$ could only fail to be the conjugate of $a + bi$ when either $a$ or $b$ is a non-real complex number. As a matter of fact $a - bi$ does fail to be the conjugate of $a + bi$ in precisely these cases. Let $a = a_0 + a_1 i$, $b = b_0 + b_1 i$ ($a_0$, $a_1$, $b_0$, $b_1$ real). Then

$$a + bi = (a_0 - b_1) + (a_1 + b_0)i \quad \text{(in standard form)}$$

$$a - bi = (a_0 + b_1) + (a_1 - b_0)i \quad \text{(in standard form)}$$

The conjugate of $a + bi$ is

$$\overline{a + bi} = (a_0 - b_1) - (a_1 + b_0)i \quad \text{(not in standard form!)}$$

and this equals $a - bi$ if and only if

$$a_0 - b_1 = a_0 + b_1 \text{ and } -(a_1 + b_0) = a_1 - b_0,$$

i.e., if and only if $b_1 = 0$ and $a_1 = 0$.

(As for examples, $a = i$, $b = 0$ is one; indeed any pair $a$, $b$ in which at least one of $a$, $b$ is a non-real complex number.)

6. Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$, $x_1, x_2, y_1, y_2$ real. Then

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i,$$

and $$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i.$$

These are real, respectively, if and only if

$$y_1 + y_2 = 0, \quad x_1 y_2 + x_2 y_1 = 0.$$

Eliminate $y_2$: $\quad x_1(-y_1) + x_2 y_1 = 0$, or $(x_1 - x_2)y_2 = 0$. Now one or other of the factors $x_1 - x_2$, $y_2$ must be zero. If $y_2 = 0$, then $y_1 = 0$ too and $z_1$, $z_2$ are both real. But if $x_1 - x_2 = 0$, then $x_2 = x_1$ while $y_2 = - y_1$; in this case $z_1 = \bar{z}_2$.

7. Formula 5-5d: $\overline{(\bar{z})} = z$. Let $z = a + bi$, in standard form. Then $\bar{z} = a - bi$ or, in standard form, $\bar{z} = a + (-b)i$. Now $\overline{(\bar{z})} = a - (-b)i$, but this is just $z = a + bi$.

Formula 5-5e: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, $a_1, a_2. b_1, b_2$ all real. Then $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ and $\overline{z_1 + z_2} = (a_1 + a_2) - (b_1 + b_2)i = (a_1 + a_2) + (-b_1 + (-b_2))i = \overline{z_1} + \overline{z_2}$. Q. E. D.

Formula 5-5f. With the same notation as above: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$, so $\overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - (a_1 b_2 + a_2 b_1)i = (a_1 a_2 - (-b_1)(-b_2)) + (a_1 (-b_2) + a_2(-b_1))i = \bar{z}_1 \bar{z}_2$.

8. Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$, $x_1, x_2, y_1, y_2$ real. Then $|z_1 - z_2|^2 + |z_1 + z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_1 + x_2)^2 + (y_1 + y_2)^2 = 2x_1^2 + 2y_1^2 + 2x_2^2 + 2y_2^2 = 2(x_1^2 + y_1^2) + 2(x_2^2 + y_2^2) = 2|z_1|^2 + 2|z_2|^2$.

9. $|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |x| \geq x$, and similarly for y. Note: $\sqrt{x^2 + y^2} \geq \sqrt{x^2}$ since $x^2 + y^2 \geq x^2$, $y^2$ being non-negative. $|x| \geq x$ since either $|x| = x$, or $|x|$ is positive while x is negative.)

10. $|x| + |y| = \sqrt{2}\,|z| = \sqrt{2}\,\sqrt{|x|^2 + |y|^2}$ if and only if $(|x| + |y|)^2 = 2(|x|^2 + |y|^2)$, if and only if $|x|^2 + 2|x|\cdot|y| + |y|^2 = 2(|x|^2 + |y|^2)$, if and only if $2|x|\cdot|y| = |x|^2 + |y|^2$. In our last condition $|x| = 0$ if and only if $|y| = 0$, i.e., $|x| = |y|$. Suppose $|x| \neq 0$ and let $r = \frac{|y|}{|x|}$. Then $2|x|\cdot|y| = |x|^2 + |y|^2$ and $|x| \neq 0$ if and only if $2r = 1 + r^2$. But $2r = 1 + r^2$ if and only if $r^2 - 2r + 1 = (r - 1)^2 = 0$, i.e., if and only if $r = \frac{|y|}{|x|} = 1$. Q.e.d.

*11. The inequality we hope to prove,

$$|x| + |y| \leq \sqrt{2}\,|z|$$

is equivalent to

$$(|x| + |y|)^2 \leq 2\,|z|^2$$

or $\quad |x|^2 + 2|x||y| + |y|^2 \leq 2|z|^2 = 2(|x|^2 + |y|^2)$

or $$2|x||y| \leq |x|^2 + |y|^2$$

but the last inequality is equivalent to

$$|x|^2 - 2|x|\cdot|y| + |y|^2 = (|x| - |y|)^2 \geq 0.$$

Now read all this backwards, and you have the proof.

*12 (b) 1) holds since x - y is a real number, so its square is non-negative. Thus $x^2 - 2xy + y^2 \geq 0$ and, adding 2xy to each side, $x^2 + y^2 \geq 2xy$.

2) is obtained by the substitution indicated.

3) $a^2c^2 + b^2d^2$ added to each side

4) factor each side

5) First inequality: extract non-negative square roots of each side. Second inequality: a real number never exceeds its absolute value.

6) Multiply by the (positive real) number 2.

7) Add $a^2 + b^2 + c^2 + d^2$ to each side

8) Factor again.

9) Make the substitution indicated and use formulas 5-5e and 5-5g.

10) Extract non-negative square roots.

(c) With $b = d = 0$, the steps become

2) $0 \geq 0$

3) $a^2c^2 \geq a^2c^2$

4) $a^2c^2 \geq (ac)^2$

5) $\sqrt{a^2c^2} \geq |ac| \geq ac$

6) $2\sqrt{a^2c^2} \geq 2ac$

7) $a^2 + 2\sqrt{a^2c^2} + c^2 \geq a^2 + 2ac + c^2$

8) $(\sqrt{a^2} + \sqrt{c^2})^2 \geq (a + c)(a + c)$

9) $(|a| + |c|)^2 \geq |a + c|^2$

10) $|a| + |c| \geq |a + c|$.

(d) Working backwards.

*13.
$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= |z_1|^2 + 2x_3 + |z_2|^2 \\ &\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 = (|z_1| + |z_2|)^2, \end{aligned}$$

Since $|\bar{z}_2| = |z_2|$. Extracting non-negative square roots:

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(Parenthetical "why?": $z_1\bar{z}_2 + \bar{z}_1 z_2 = z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}$ by formulas 5-5f and 5-5d. So formula 5-5b gives $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2x_3$, since $x_3$ is the real part of $z_1\bar{z}_2$.)

*14. First Proof. Consider

$$|z_1| = |(z_2 - z_1) - z_2| = |(z_2 - z_1) + (-z_2)|$$
$$\leq |z_2 - z_1| + |-z_2| = |z_2 - z_1| + |z_2|,$$

by the inequality for sums (Exercises *12, *13). Thus, subtracting $|z_2|$ from each side,

(*) $$|z_1| - |z_2| \leq |z_2 - z_1|.$$

If the left member is non-negative, it equals its absolute value, so

$$\big||z_1| - |z_2|\big| \leq |z_2 - z_1|.$$

If it is negative then $|z_2| - |z_1|$ is positive. If we interchange $z_1$ and $z_2$ in (*) we get

$$|z_2| - |z_1| \leq |z_1 - z_2| = |z_2 - z_1|,$$

so again we get

$$\big||z_1| - |z_2|\big| \leq |z_2 - z_1|$$

since $\big||z_1| - |z_2|\big| = \big||z_2| - |z_1|\big|$ .

Second proof (in the manner of the solution to Exercise *13):

$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= z_1\bar{z}_1 - (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2 \\ &= |z_1|^2 - 2x_3 + |z_2|^2 \\ &\geq |z_1|^2 - 2|z_1\bar{z}_2| + |z_2|^2 \end{aligned}$$

so $|z_1 - z_2|^2 \geq (|z_1| - |z_2|)^2$. Extracting non-negative square roots: $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ .

Exercises 5-6 Answers

1. (a) $2 + 0i$
   (b) $2 + 0i$
   (c) $0 + 2i$
   (d) $0 + 2i$
   (e) $0 + \sqrt{2}\, i$
   (f) $0 + \sqrt{\pi}\, i$

2. (a) $|c| + 0i$
   (b) $|c| + 0i$
   (c) $0 + |c|i$
   (d) $0 + |c|i$

*3. The proof given in Chapter 1 requires a and b to be non-negative real numbers. Thus "this situation" does not contradict the result in Chapter 1. When $a$ and $b$ are negative real numbers--as in this exercise--we are obliged to use Definition 5--and write $\sqrt{a}\,\sqrt{b} = (i\sqrt{|a|})(i\sqrt{|b|}) = -\sqrt{|a|}\,\sqrt{|b|}$ before multiplying the radicands, since the radicands must be non-negative if we are to use the formula from Chapter 1. [Let us examine a proof of the formula from Chapter 1 to see where it breaks down if $a$ and $b$ fail to be non-negative real numbers. Suppose, then, that $a$ and $b$ are any real numbers. Set

$$u = \sqrt{ab}, \quad v = \sqrt{a}, \quad w = \sqrt{b}.$$

Then $u^2 = ab$, $v^2 = a$, $w^2 = b$. Multiplying the last two equalities we get $(vw)^2 = ab$. Thus $vw$, or $\sqrt{a}\,\sqrt{b}$, is one of the (two) solutions of $x^2 = ab$. The two solutions of the last equation are $u$ and $-u$. If $a$ and $b$ are both non-negative (as in Chapter 1), then so is $ab$, and the three numbers $u$, $v$, $w$ are all real and non-negative. In this case we can identify the roots $u$ and $vw$ of $x^2 = ab$, proving the formula from Chapter 1. If $a$ and $b$ are both negative, then $ab$ is positive. In this case

$v$ and $w$ are pure imaginary numbers, $i\sqrt{|a|}$ and $i\sqrt{|b|}$, respectively, while $u$ is a positive real number. In this case the product $vw$ is a negative real number. Thus $vw = -u$. Curiously, if only one of $a$, $b$ is negative while the other is non-negative, we can still prove the formula from Chapter 1. Suppose $a < 0$, $b \geq 0$. Then $ab \leq 0$, and $\sqrt{a}\ \sqrt{b} = (i\sqrt{|a|})\ \sqrt{|b|} = i\sqrt{|a|\,|b|} = i\sqrt{|ab|} = \sqrt{ab}$. Thus the formula $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$ breaks down only when $a$ and $b$ are both negative.]

4. $0 + 1i$; $0 + (-1)i$
5. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $-\frac{1}{2} + (-\frac{\sqrt{3}}{2})i$
6. $-1 + i$; $-1 + (-1)i$
7. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $\frac{1}{2} + (-\frac{\sqrt{3}}{2})i$
8. $-\frac{1}{3} + \frac{\sqrt{11}}{3}i$; $-\frac{1}{3} + (-\frac{\sqrt{11}}{3})i$
9. $-2 + 2i$; $-2 + (-2)i$
10. $2 + 2i$; $2 + (-2)i$
11. $-\frac{1}{4} + \frac{\sqrt{7}}{4}i$; $-\frac{1}{4} + (-\frac{\sqrt{7}}{4})i$
12. $2(1 + \sqrt{1 + 2a}) + 0i$; $2(1 - \sqrt{1 + 2a}) + 0i$
13. $-\frac{1}{2m} + \frac{\sqrt{3}}{2|m|}i$; $-\frac{1}{2m} + (\frac{-\sqrt{3}}{2|m|})i$
14. $0 + 2i$; $0 + (-1)i$
15. If $\frac{c}{a} \geq 0$: $0 + \sqrt{\frac{c}{a}}\,i$; $0 + (-\sqrt{\frac{c}{a}})i$.

    If $\frac{c}{a} \leq 0$: $\sqrt{-\frac{c}{a}} + 0i$; $-\sqrt{-\frac{c}{a}} + 0i$.
16. $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$

    $z^3 - 8 = 0$ if and only if $z - 2 = 0$ or $z^2 + 2z + 4 = 0$

    $z_1 = 2$; $z_2 = -1 + \sqrt{3}\,i$; $z_3 = -1 - \sqrt{3}\,i$

17. $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi = 0 + i.$

$$x^2 - y^2 = 0, \quad 2xy = 1.$$

$$x_1 = \frac{1}{\sqrt{2}}, \quad y_1 = \frac{1}{\sqrt{2}}; \quad x_2 = -\frac{1}{\sqrt{2}}, \quad y_2 = -\frac{1}{\sqrt{2}}.$$

$$z_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}; \quad z_2 = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}.$$

18. $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi = 0 - i$

$$x^2 - y^2 = 0, \quad 2xy = -1.$$

$$x_1 = \frac{1}{\sqrt{2}}, \quad y_1 = -\frac{1}{\sqrt{2}}; \quad x_2 = -\frac{1}{\sqrt{2}}, \quad y_2 = \frac{1}{\sqrt{2}}.$$

$$z_1 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}; \quad z_2 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

19. _If_ we suppose that $a$, $b$, $c$ are real numbers, we may say, using Theorem 5-7, that one of the numbers

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

is $z_1$ while the other is $z_2$. Thus

$$z_1 + z_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - b}{2a} = -\frac{b}{a}$$

and

$$z_1 z_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

[However, Exercise 19 does not explicitly announce that $a$, $b$, $c$ are real numbers. We give another solution, equally valid whether or not we make the assumption that the coefficients are real. We have two cases: (1) $z_1$ and $z_2$ are different roots, (2) $az^2 + bz + c = 0$, $a \neq 0$ has only one root. For _case (1)_ consider the pair of true statements

$$az_1^2 + bz_1 + c = 0$$
$$az_2^2 + bz_2 + c = 0.$$

Subtracting, we get

$$a(z_1^2 - z_2^2) + b(z_1 - z_2) = (z_1 - z_2)\ [a(z_1 + z_2) + b] = 0.$$

Since $z_1 \neq z_2$, we have $z_1 - z_2 \neq 0$, hence

$$a(z_1 + z_2) + b = 0 \quad \text{or} \quad \underline{z_1 + z_2 = -\frac{b}{a}}.$$

Adding, we get

$$a(z_1^2 + z_2^2) + b(z_1 + z_2) + 2c = 0,$$

or $a[(z_1 + z_2)^2 - 2z_1z_2] + b(z_1 + z_2) + 2c = 0.$

Hence, since $z_1 + z_2 = -\frac{b}{a}$, as we have just seen,

$$a[(-\frac{b}{a})^2 - 2z_1z_2] + b(-\frac{b}{a}) + 2c = 0,$$

or $\frac{b^2}{a} - 2az_1z_2 - \frac{b^2}{a} + 2c = 0,$

or $2az_1z_2 = 2c$. Thus $\underline{z_1z_2 = \frac{c}{a}}$.

For Case (2), we first prove the following lemma: If the equation $z^2 = A$, where $A$ is a complex number, has only one solution, then $A = 0$. Proof: If $z_1$ belongs to the solution set of $z^2 = A$, then $-z_1$ also belongs to the solution set; for $(-z_1)^2 = z_1^2 = A$ if $z_1^2 = A$. If the solution set contains only one element, then $z_1 = -z_1$, so $z_1 = 0$. Hence $A = z_1^2 = 0$ too. Q.e.d. Now, suppose $az^2 + bz + c = 0$, $a \neq 0$, has only one solution $z_1$. Completing the square

$$(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Since this equation can have only one solution if the equivalent equation $az^2 + bz + c = 0$ has only one solution, we conclude that $\frac{b^2 - 4ac}{4a^2} = 0$. The single solution of either equation is therefore given by $(z_1 + \frac{b}{2a})^2 = 0$, so $2z_1 = - \frac{b}{a}$, which is part of what we wanted to prove. But $b^2 - 4ac = 0$, so $(- \frac{b}{a})^2 = 4 \frac{c}{a}$, thus $z_1{}^2 = \frac{1}{4}(- \frac{b}{a})^2 = \frac{c}{a}$ and this is the rest of what we wanted to prove.]

20. We use the results of Exercise 19:

$$\begin{aligned} a(z - z_1)(z - z_2) &= a[z^2 - (z_1 + z_2)z + z_1z_2] \\ &= a(z^2 + \frac{b}{a} z + \frac{c}{a}) \\ &= az^2 + bz + c. \end{aligned}$$

21. (a) $z_1 + z_2 = 2$, $z_1z_2 = 2$: $z^2 - 2z + 2 = 0$.

(b) $z_1 + z_2 = 2 + 2i$, $z_1z_2 = - 1 + 2i$: $z^2 - (2 + 2i)z - 1 + 2i = 0$.

(c) $z_1 + z_2 = 0$, $z_1z_2 = 0$: $z^2 - 0z + 0 = 0$, or $z^2 = 0$.

(d) $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$, $z_1z_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$: $z^2 - [(a_1 + a_2) + (b_1 + b_2)i]z + [(a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i] = 0$.

*22. Extending the idea in Exercise 20;

$$[z - (1 + 2i)]\ [z - (1 - i)][z - (1 + i)] = 0,$$

or--multiplying out the left member--

$$z^3 - (3 + 2i)z^2 + (4 + 4i)z - (2 + 4i) = 0.$$

There is no quadratic equation having all three roots, for the formula in Exercise 20 shows that no quadratic equation may have more than two roots: If $az^2 + bz + c = a(z - z_1)(z - z_2) = 0$, $a \neq 0$, then either $z - z_1 = 0$ or $z - z_2 = 0$; i.e., $z = z_1$ or $z = z_2$. Moreover no quadratic expression such as $az^2 + bz + c$ can be written as a product of three first degree factors, say $(z - z_1)(z - z_2)(z - z_3)$, times a constant: For any such product produces a $z^3$ - term and no quadratic can have such a term.

Exercises 5-7 Answers

1, ..., 18 the graphs are omitted.

19. (First solution.) O, A, B collinear if and only if $b:a = d:c$. C is on the same line if and only if $(b + d):(a + c) = b:a$. (The equivalence of this pair of proportions should be familiar to the student from his study of geometry. Note on the colon notation: We use the notation $b:a = d:c$ to mean (1) $\frac{b}{a} = \frac{d}{c}$, provided neither a nor c is zero, but, in additon, (2) that $a = 0$ if and only if $c = 0$.)

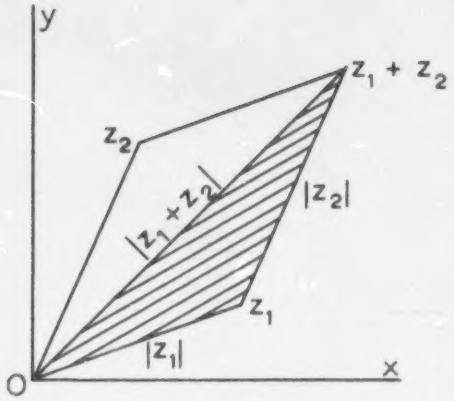
(Second solution.) We note first that $y_0x - x_0y = 0$ is an equation for the line joining $(0, 0)$ and $(x_0, y_0)$--whether or not one of $x_0$, $y_0$ be zero. Thus the following are equations of the lines indicated

$$bx - ay = 0 \quad : \quad OA,$$
$$dx - cy = 0 \quad : \quad OB,$$
$$(b + d)x - (a + c)y = 0 \quad : \quad OC.$$

We are to show that any point on both of OA, OB is also on OC. Let $(x_1y_1)$ be a point on OA and on OB. Then

$$bx_1 - ay_1 = 0 \quad \text{and} \quad dx_1 - cy_1 = 0.$$

Adding, $(b + d)x_1 - (a + c)y_1 = 0$, so $(x_1y_1)$ is also on C.



20. The shaded triangle has sides of length $|z_1|$, $|z_2|$, $|z_1 + z_2|$. Since the sum of the lengths of two sides of a triangle exceeds the length of the third side, $|z_1 + z_2| < |z_1| + |z_2|$; unless the points represented by 0, $z_1$, $z_2$ be collinear. If 0, $z_1$, $z_2$ are collinear, $|z_1 + z_2| = |z_1| + |z_2|$. Hence $|z_1 + z_2| \leq |z_1| + |z_2|$, for any pair $z_1$, $z_2$ of complex numbers.

21. $|z| = 1$ if and only if $\sqrt{x^2 + y^2} = 1$, for $z = x + yi$, in standard form. But $\sqrt{x^2 + y^2} = 1$ if and only if $x^2 + y^2 = 1$ (since $\sqrt{x^2 + y^2}$ represents the non-negative root of $x^2 + y^2$). Thus the set of points corresponding to complex numbers $z$ satisfying $|z| = 1$ is exactly the same as the set of points whose rectangular coordinates $(x, y)$ satisfy $x^2 + y^2 = 1$. The latter set of points consists of all those lying on the circle with center (0, 0) and radius 1. Since $|z|$ is the distance from the origin to the point represented by $z$, $|z| < 1$ holds for all those points--and only those points--interior to the "unit circle"; similarly $|z| > 1$ describes the points exterior to the "unit circle".

22. $az + b = 0$ implies $z = -\frac{b}{a}$, for $a \neq 0$. But $0 < a < b$ implies $|z| = |-\frac{b}{a}| = \frac{b}{a} > 1$. Cf. Exercise 21.

*23. Let $z_1$, $z_2$ be the roots of $az^2 + bz + c = 0$. There are two cases: Case (1) $b^2 - 4ac \leq 0$; so that the roots are either real and equal, or are not real but are complex conjugates. In this case $|z_1| = |z_2|$. Case (2): $b^2 - 4ac > 0$ in this case the roots are real and distinct.

Case (1). $|z_1| = |z_2|$. From $z_1 z_2 = \frac{c}{a} > 1$ we have $|z_1|^2 = |z_2|^2 > 1$; i.e., $|z_1| > 1$ and $|z_2| > 1$.

Case (2). (First solution.) Since $z_1 + z_2 = -\frac{b}{a} < 0$, at least one of the roots is negative. But since $z_1 z_2 = \frac{c}{a} > 0$, both are negative. From $0 < a < b < c$, we get $|z_1| + |z_2| = |z_1 + z_2| = \frac{b}{a} < \frac{c}{a} = |z_1| \cdot |z_2|$. Hence

$$|z_1| < |z_1| \cdot |z_2| - |z_2| = |z_2| \{|z_1| - 1\}$$

But $|z_1| \neq 0$ (for $\frac{c}{a} > 0$) so $\{|z_1| - 1\} > 0$ and $|z_1| > 1$. Similarly $|z_2| > 1$.

Case (2). (Second solution.) $b^2 - 4ac > 0$. Here we must have $b > 2a$, for $b^2 > 4ac > 4a^2$ since $c > a$. Hence

$$|z_2| = \left| \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right| = \frac{b + \sqrt{b^2 - 4ac}}{2a} > \frac{b}{2a} > 1.$$

Now $$|z_1| = \left| \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right| = \left| \frac{b - \sqrt{b^2 - 4ac}}{2a} \right| .$$

Since $b^2 > b^2 - 4ac$, we have $b > \sqrt{b^2 - 4ac}$, so

$$|z_1| = \frac{b - \sqrt{b^2 - 4ac}}{2a}$$

Now $|z_1| > 1$ if and only if $b - \sqrt{b^2 - 4ac} > 2a$, or $b - 2a > \sqrt{b^2 - 4ac}$. The last inequality is equivalent to $(b - 2a)^2 = b^2 - 4ab + 4a^2 > b^2 - 4ac$,

$$4ac - 4ab + 4a^2 > 0,$$

or $$c - b + a > 0.$$

But $$c - b + a > b - b + a = a > 0.$$

Exercises 5-8 Answers.

1. If $z_1$ and $z_2$ are roots of a quadratic equation with real coefficients, then the equation is of the form
$$a(z - z_1)(z - z_2) = 0,$$
where $a$ is a real number, not zero. Thus
$$a[z^2 - (z_1 + z_2)z + z_1z_2] = 0,$$
$$a(z^2 - z + 4) = 0.$$

2. If $z = a + bi$, in standard form, then $z^2 = (a^2 - b^2) + 2abi$ and its real part is $a^2 - b^2$. We want this real part zero, so $a^2 = b^2$. Thus $z$ has the form $a + ai$ or $a - ai$ ($a$ real). Conversely, every such $z$ has the property that the real part of its square is zero.

3. If $z = a + bi$, in standard form, and $z \neq 0$, then $\bar{z} = a - bi$ and $\frac{1}{\bar{z}} = \frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2} i$. The line joining $(0, 0)$ and $(a,b)$ has the equation $ay - bx = 0$ and clearly $\left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}\right)$ lies on this line.

4. The equation is equivalent to $|z| = 2$. Hence the locus is a circle with center at the origin and radius 2.

5. Let $z_1 = a + bi$, $z_2 = c + di$, in standard form. Then $\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i$. The equation of the line through (0, 0) and (a, b) is $ay - bx = 0$, and $z_2$ will lie on this line if and only if $ad = bc$. But this is equivalent to the condition that the imaginary part of $\frac{z_1}{z_2}$ be zero.

6. If $z = a + bi$, in standard form, then our first condition gives $a = b$. But $\sqrt{a^2 + b^2} = \sqrt{2a^2} = 1$ gives $a^2 = \frac{1}{2}$. The $z$ in question are therefore $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and $\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.

7. $z \bar{z} = |z|^2 = 1$, hence $\bar{z} = \frac{1}{z}$, since the multiplicative inverse is unique.

8. $\frac{1}{1 + i} = \frac{1 - i}{2}$, $\overline{1 + i} = 1 - i$, $|1 + i| = 2$.

9. $\frac{1}{2 + 3i} = \frac{2 - 3i}{13}$, $\overline{2 + 3i} = 2 - 3i$, $|2 + 3i| = 13$.

10. $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$. To prove this let $z' = \frac{\bar{z}}{|z|^2}$, then

$zz' = \frac{z \cdot \bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$, hence $z' = \frac{\bar{z}}{|z|^2} = \frac{1}{z}$. (Cf. Section 5-5, formula 5-58.)

11. Let $D$ be the discriminant of the equation $rz^2 + (1 + r)z + 2 = 0$. Then $D = (1 + r)^2 - 4(2r) = r^2 - 6r + 1$. The equation has non-real complex roots if and only if $D < 0$, i.e., $-3 - 2\sqrt{2} < r < -3 + 2\sqrt{2}$. It has a single root if and only if $D = 0$, i.e., $r = -3 - 2\sqrt{2}$ or $r = -3 + 2\sqrt{2}$.

12. If $3 + 2i$ is a root, then so is $\overline{3 + 2i} = 3 - 2i$ if we are to have real coefficients. Hence the equation has the form

(a real, not zero)

$$a[z - (3 + 2i)][z - (3 - 2i)] = 0$$

$$a(z^2 - 6z + 13) = 0.$$

13. $z^4 = - 1$ if and only if $z^2 = i$ or $z^2 = -i$. Using the results of Exercises 17, 18, Section 5-6, the four roots are

$$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}; \quad - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}; \quad \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}; \quad - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

*14. $i > 0$ implies that $i^2 > 0$ or $-1 > 0$, which is impossible. Similarly $i < 0$ implies $i^2 > 0$, also impossible.

# Chapter 6

## Introduction

The fundamental aim of this chapter is algebraic: to study the solution sets of equations and inequalities of the second degree, individually and in systems. Nevertheless, we begin geometrically with a systematic approach to the conic sections. In earlier chapters the equations of the circle and the parabola have been encountered, but not in the context of a systematic geometrical study. Here we provide geometrical definitions for the conic sections, and from these derive their equations. The reason for a geometrical approach to an algebraic subject is twofold: first, because the geometry of the conics is interesting to the students; second, because we wish to establish a strong association between algebraic expressions and algebraic systems and their geometric counterparts.

The treatment of the conics has been reduced to what are considered the minimal essentials, in order to get on with the main algebraic task. Thus such topics as the focus-directrix definitions of ellipse and hyperbola, eccentricity and latus rectum are not mentioned.

The main emphasis in the latter part of the chapter is on the equivalence of systems of equations as a device for obtaining their solutions. This is an idea with wide implications. It will appear again in Chapter 7. It leads directly into the treatment of systems involving more than 3 equations or inequalities, which are of growing importance both in pure mathematics and in such modern applications as the theory of games and linear programming.

The basic device employed in this chapter to obtain equivalent systems from a given system of equations is what we call the principle of linear combination. This term is best explained by considering as an example the system

$$x^2 + y^2 - 17 = 0$$
$$2x^2 + y^2 - 18 = 0.$$

In Section 6-8 we show that this system is equivalent to the system consisting of either of the above equations together with an equation of the form

$$a(x^2 + y^2 - 17) + b(2x^2 + y^2 - 18) = 0,$$

where neither $a$ nor $b$ is zero. The expression on the left of this equation is called a linear combination of the two expressions in parentheses; hence the name given to the principle.

Many of the mathematical ideas of this chapter are far-reaching and important, as well as intrinsically interesting. It is fortunate that ideas of wide scope can here be related to material that is well within the capability of students at this educational level.

## Comments on Specific Sections.

### 6-1 to 6-4. Conic sections.

The first three sections are primarily concerned with the derivation of equations for the parabola, ellipse and hyperbola. We have chosen to derive equations for particular numerical examples, rather than for the general case. This is partly because we are not going to make subsequent use of the general formulas, and partly because we believe that the more able students should be able to derive the general equation after having seen the derivation for a particular case.

It will be noticed that in Chapter 3 the graph of a quadratic function was identified as a parabola, while in the present chapter the parabola has been defined by means of the focus-directrix property. Here are two apparently different definitions which should be reconciled. The "locus" definition of the parabola, in terms of directrix and focus, is essentially the one that was used by Greek mathematicians several centuries B.C. With the introduction of coordinate geometry in the seventeenth century, it became important to enquire what curve one would obtain if one began with an arbitrary equation; conversely, mathematicians sought for the algebraic equations associated with well-known curves. In Chapter 3 we are investigating the geometry of the quadratic function, and are led to consider graphs which are actually parabolas with vertical axes. In Chapter 6, conversely, we begin from the Greek point of view, with the locus definition of the conics, and find the equation of the parabola by first specifying the orientation of the directrix and focus with regard to the coordinate axes. In both chapters we are dealing with facets of one great fact of coordinate geometry: the correspondence between the conic sections and the general quadratic equation. In Chapter 3 we study this correspondence beginning from the algebraic viewpoint, while in Chapter 6 we approach it from the geometric viewpoint.

It is worthwhile here to mention the so-called degenerate conics, because they may arise as forms of the general "second degree graph." We list them below, with a simple illustration of each:

| | |
|---|---|
| A single point | $x^2 + 2y^2 = 0$ |
| Two intersecting straight lines | $x^2 - 2y^2 = 0$ |
| Two parallel straight lines | $x^2 = 9$ |
| Two coincident straight lines | $x^2 = 0$ |
| No locus | $x^2 + 2y^2 = 5$, or $x^2 = -9$ |

It is interesting to note that the locus definition which we have used for the ellipse can be derived from the definition of the ellipse as the curve in which a plane cuts a circular cone. Suppose we have a paper cone, such as is used as a drinking cup. We drop into it two spheres: a large marble and a tennis ball; we shall assume that their sizes are such that the two spheres do not touch each other. Of course, each sphere touches the cone along a circle of contact. It can be shown that the planes of these circles are parallel, and that they are both perpendicular to the axis of the cone. A line passing through the vertex and lying on the surface of the cone is called a generator of the cone. Any generator will cut the two circles of contact of the spheres at points $Q$ and $Q'$; it can be shown that the distance $d(Q,Q')$ is constant while the generator traces out the cone.

Let the paper cone, the marble and the tennis ball now be replaced by a mathematical cone and two mathematical spheres.

Consider any plane which lies between the two spheres and touches them both. Suppose it touches the small sphere at F, and the large sphere at F'. The plane will cut the cone in a curve. We wish to

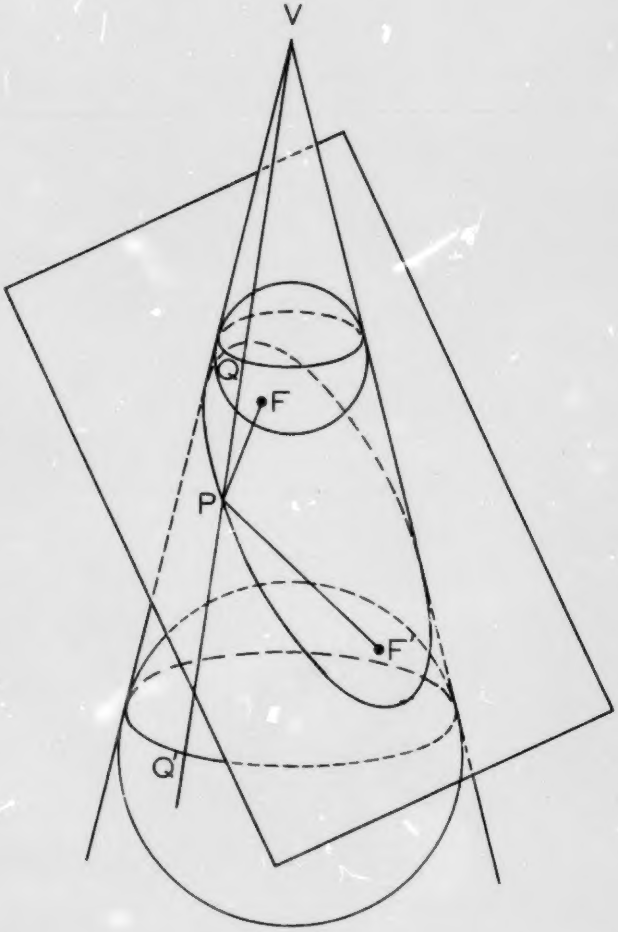


Figure T 6-1

show that if P is any point on the curve, then d(P,F) + d(P,F') = a constant, so that the curve is an ellipse as defined by the "string property."

The proof will depend on one fact from solid geometry: if P is a point external to a sphere, and if PM and PN are tangents to the sphere, M and N being points of contact, then d(P,M) = d(P,N).

Now let P be any point on the curve in which the plane cuts the cone. Join P to the vertex V. The line PV will cut the circles of contact of the two spheres in points Q and Q'; we will suppose that Q is on the small sphere and Q' is on the large one. Join P to the points F and F' at which the plane touches the spheres. Now d(P,F) = d(P,Q), since the lines PF and PQ are both tangent to the small sphere, and F and Q are the points of contact. Similarly d(P,F') = d(P,Q'), since PF' and PQ' are both tangents to the large sphere. Thus

$$d(P,F) + d(P,F') = d(P,Q) + d(P,Q').$$

But P is between Q and Q' on the line PV, so that d(P,Q) + d(P,Q') = d(Q,Q'), which is constant.

It has been pointed out that if the axis of a conic is not parallel to one of the coordinate axes, then an xy term will appear in its equation. To illustrate this fact, we consider a parabola whose focus is at F(2,2) and whose directrix is x + y = 0. Let P(x, y) be a point on the locus. Then if Q is the foot of the perpendicular from P to the directrix, we have d(P,F) = d(P,Q). In order to express this relation in algebraic form we need to find the

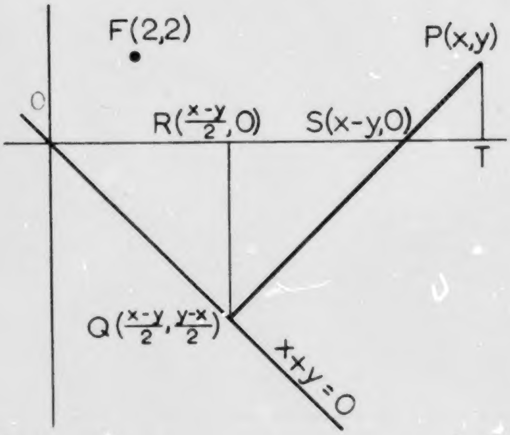


Fig. T 6-2

coordinates of Q. For this purpose we first find the coordinates of S, the point where PQ cuts the x-axis. Since PTS is an isosceles right triangle, $d(S,T) = d(T,P) = y$, and hence S has coordinates $(x - y, 0)$. Thus R, the midpoint of OS, has coordinates $(\frac{x-y}{2}, 0)$, and it immediately follows that the coordinates of Q are $(\frac{x-y}{2}, \frac{y-x}{2})$. Hence the equation $d(P,F) = d(P,Q)$ can be written in the form

$$\sqrt{(x-2)^2 + (y-2)^2} = \sqrt{(x - \frac{x-y}{2})^2 + (y - \frac{y-x}{2})^2}$$

or, squaring both sides and simplifying,

$$x^2 - 2xy + y^2 - 8x - 8y + 16 = 0.$$

Two observations should now be made:

(a) the presence of the xy term is a consequence of the fact that the directrix is not parallel to either axis;

(b) the second degree terms form a perfect square. This will always be the case for the equation of a parabola (including the degenerate forms of parabola: two parallel, coincident or imaginary straight lines).

In more complete discussions of the conic sections, it is proved that the graph of the general second degree equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

may be partially identified as to type by means of the quantity $b^2 - 4ac$, as indicated in the following table:

| $b^2 - 4ac$ | Type of graph |
|---|---|
| 0 | Parabola; two straight lines (real and distinct, real and coincident); or no locus. |
| $> 0$ | Hyperbola; two real intersecting straight lines. |
| $< 0$ | Ellipse; point ellipse; or no locus. |

It may be verified that the condition $b^2 - 4ac = 0$ is equivalent to the requirement that the second degree terms $ax^2 + bxy + cy^2$ form a perfect square.

The quantity $b^2 - 4ac$ is one of several quantities associated with the conic sections, known as invariants. This means that if the quantity is calculated for a given conic, and if the given conic is then subjected to a translation or rotation of axes, and the quantity is again calculated for the new equation, its value will be unaltered.

## 6-5 Asymptotes.

The essential property of the asymptote, as developed in connection with our discussion of the asymptotes of a hyperbola, is that the distance, measured in a specified direction, from a point on the hyperbola to the asymptote approaches zero as a limit, as the distance along the asymptote, measured from the center of the hyperbola, becomes infinite. The idea of a limit is complex, and a full treatment comes much later in the student's mathematical experience. Nevertheless, an early start needs to be made in dealing with ideas of this kind. The rudimentary notion of a limit is unavoidably involved when we consider such matters as the length and area of a circle, the idea of the instantaneous velocity of a car, and the concept of a tangent to a curve. Finally, the student will not achieve a satisfactory concept of real numbers until he has gained some mastery of the limit concept. In fact, it may be said that the development of the real number concept and the development of the limit concept must necessarily go forward hand in hand, for neither can progress very far without the other.

No treatment of the hyperbola can be complete without some consideration of the asymptotes. This is the real reason the concept is introduced in this chapter. The limit concept, which is introduced in connection with the asymptotes, will be taken up again in Chapter 11, where it appears in connection with summation of series. From the remarks made above, it should be clear that the limit concept is a unifying idea in the student's mathematical experience from its beginning in grade school arithmetic and geometry, right through to the most advanced mathematics with which the student may have to deal.

### 6-7, 6-8. Equivalent systems of equations.

Equivalent systems are emphasized for two reasons. First, with their use one gains a really deep insight into the mathematics of the processes employed, with the result that genuine mathematical understanding is fostered, rather than dependence on rules. Second, the central idea is one which will continue to be fruitful in the student's future experience, no matter how far he goes in mathematics.

Note particularly that Chapter 7 makes use of the same definition of equivalent systems as is used in this chapter: two systems are equivalent if they possess the same solution set. The method of triangulation developed in Chapter 7 is essentially a method of obtaining, from a given system of equations, a second system which is so simple that its solution (or solution set) may be immediately obtained.

Exercises 6-1. Comments.

The parts of the parabola can be identified and related to previous work with quadratic functions.

Example 1-2 develops the equation of a parabola for a special case, with the general equation form appearing in the Exercises. In sketching the curve, the symmetry can be exploited in forming the coordinate pairs for symmetric points.

Exercises 6-1 contain many duplicated types, so that a selection can be made. Example 1 is duplicated in Ex. 1-4 and covers the four cases. In Ex. 5-7 sketches can be made from a few points, with use of symmetry. The more capable student should be able to do Ex. 7, even though this type is considered in the next section. There may be a limited response to Ex. 8 which can be amplified by teacher or class discussion. Student understanding of the locus property is emphasized by Ex. 9. Set notation is reviewed in Ex. 10-1. A short sketching method can be obtained from Ex. 13-16 by use of the coordinates of the vertex and coordinates of the end points of the focal chord. The quadratic function is related to the parabola by Ex. 17.

Exercises 6 - 1. Answers.

1. $y^2 = 12x$
2. $y^2 = -16x$
3. $x^2 = -20y$
4. $x^2 = 24y$
5. Parabola with vertex at (0,0) and opening upward about the y-axis
6. (Make sketches for each)

*7. ( " " " " )

8. Cross section of automobile headlight, reflector, cross section of radar antenna, trajectories of missiles, suspension bridge curve, curve of a suspended wire, curve of a necklace suspended from the clasp. The last two are catenaries, but are approximately parabolas.
9. Assume $P_1(x_1, y_1)$ as a point satisfying $y^2 = 8x$, i.e., $y_1^2 = 8x_1$. If $d(P_1, F)$ is the distance between $P_1$ and the focus (2, 0), then from a right triangle

$$\overline{d(P_1, F)}^2 = (x_1-2)^2 + y_1^2$$

$$= (x_1-2)^2 + 8x_1 \qquad \text{since } y_1^2 = 8x_1$$

$$= (x_1+2)^2$$

$$\therefore d(P_1, F) = x_1 + 2$$

Since the distance between $P_1$ and the directrix $x = -2$ is $[x_1 - (-2)] = x_1 + 2$, then $P_1$ is equidistant from (2,0) and the line $x = -2$

10. a. $x^2 = -4y$

    b. A parabola with vertex at (0,0), directrix as $y = 1$, and focus at (0, -1)

e. (Sketch)

11. a. $x^2 = 4y$; a parabola with vertex at (0,0), directrix as $y = -1$, and focus (0, 1) (Sketch)

b. $x^2 = 2ay$; a family of parabolas with vertices at (0,0), directrix as $y = \frac{-a}{2}$, and focus $(0, \frac{a}{2})$. (Sketch)

c. $y^2 = 2ax$; a family of parabolas with vertices at (0,0), directrix as $x = \frac{-a}{2}$, and focus $(\frac{a}{2}, 0)$. (Sketch)

12.

| | axis | vertex | opens | |
|---|---|---|---|---|
| a. | y | (0,0) | up | (sketch) |
| b. | y | (0,0) | down | (sketch) |
| c. | x | (0,0) | right | (sketch) |
| d. | x | (0,0) | left | (sketch) |
| e. | x | (0,0) | left | (sketch) |
| f. | y | (0,0) | up | (sketch) |

g. This curve is the right half of the parabola $x^2 = y$ with $x \geq 0$ and $y \geq 0$

h. This curve is the upper half of the parabola $y^2 = x$ with $x \geq 0$ and $y \geq 0$

13. $y^2 = 4ax$; if $a > 0$, then curve opens to the right; if $a < 0$, curve opens to the left

14. Curve for $y^2 = 4ax$ has focus at (a,0) from Ex. 13. Let coordinates of the end points of the focal chord be $(a, \overline{y})$ and $(a, -\overline{y})$ since focal chord is perpendicular to the axis of the parabola. These points lie on the parabola so

$\overline{y}^2 = 4a(a) = 4a^2$ and $\overline{y} = \pm 2a$. Hence, $(a, \overline{y}) = (a, 2a)$ and $(a, -\overline{y}) = (a, -2a)$.

Exercises 6-2. Comments.

Note the use of general equation form in Example 1 as the simplified form of the equation. The right member of the radical equation also can be expressed as (x - 2), the distance between two points on a horizontal line. The property of the vertex being midway between the directrix and focus will be needed later in Exercise 6-2, Ex. 2-3. It may be necessary to review the content of Chapter 3 referred to before going into Example 2. An alternate form for completing the square in Example 2 is

$$2y - 3 = x^2 + 6x$$

$$(2y - 3) + 9 = x^2 + 6x + 9$$

$$2(y + 3) = (x + 3)^2$$

The end of Example 2 is a quick jump making use of Chapter 3 material to determine the vertex coordinates and the axis of symmetry equation from the last equality. This particular form is generalized in Exercises 6-2, Ex. 2-3.

In Exercises 6-2, Ex. 1 offers choice of problems, with parts e and f being more difficult. Ex. 2-3 can be used in class discussion as a short method of sketching. In Ex. 5 two solutions are suggested, one using a geometric approach and the other algebraic. The last problem involves rather laborious computation and plotting, but with interesting results. The axis and vertex of the parabolas are to be estimated, not computed. Note that some of the curves are not parabolas.

**Exercises 6-2. Answers.**

1. a. $x^2 - 4y + 4 = 0$

   b. $y^2 - 6x - 4y + 1 = 0$

   c. $y^2 + 4x + 2y - 11 = 0$

   d. $x^2 + 2x - 10y - 4 = 0$

   e. $y^2 - 2ax + 3a^2 = 0$

   f. $y^2 - 2ax - 2ay + 4a^2 = 0$

2. Apply the locus definition of the parabola to form

$$\sqrt{[x-(a+c)]^2 + (y-b)^2} = \sqrt{[x-(a-c)]^2 + (y-y)^2}$$

Simplifying yields $(y-b)^2 = 4c(x-a)$

3. The directrix has the equation $y = b - c$, so from the locus definition of the parabola,

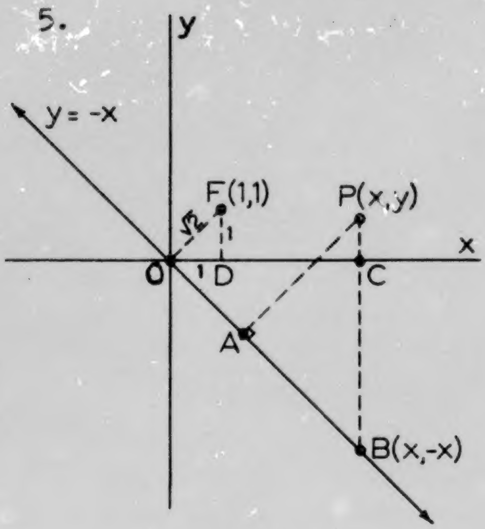
$$\sqrt{(x-a)^2 + [y-(b+c)]^2} = \sqrt{(x-x)^2 + [y-(b-c)]^2}$$

Simplifying yields $(x-a)^2 = 4c(y-b)$

4.

| | Vertex | Axis | |
|---|---|---|---|
| a. | (2, 1) | $y = 1$ | (sketch) |
| b. | (1, 7) | $y = 7$ | " |
| c. | (-7, 3) | $x = 3$ | " |
| d. | (-1, 4) | $x = -1$ | " |
| e. | (-2, -1) | $x = -2$ | " |
| f. | (a, -a) | $x = a$ | " |

5.



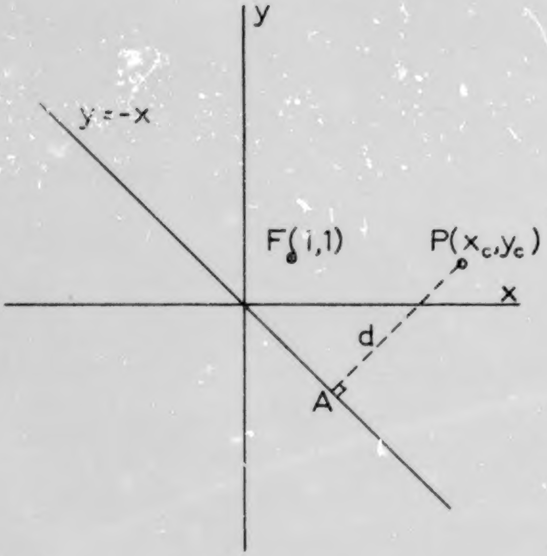
Let P(x,y) be a point on the parabola, then PC perpendicular to the x-axis determines point B having coordinates (x, -x) since B lies on the line of y = -x. These coordinates indicate that $\triangle$ OCB is right isosceles, so $\measuredangle$ B = 45°. Then $\triangle$ ABP is also right isosceles and is similar to right isosceles triangle ODF. From proportional sides of these similar triangles, $\frac{d}{\overline{PB}} = \frac{1}{\sqrt{2}}$ and since $\overline{PB} = y - (-x) = x + y$, then $d = \frac{x+y}{\sqrt{2}}$. From the locus definition of the parabola,

$$\sqrt{(x-1)^2 + (y-1)^2} = \frac{x+y}{\sqrt{2}}$$

Simplyfying yields the equation

$$x^2 - 2xy + y^2 - 4x - 4y + 4 = 0$$

An alternate solution determines $\overline{PA} = d$ from coordinates of A and P by use of the distance formula. The coordinates of A can be found by simultaneous solution of the equations for the lines of PA and y = -x. Let $P(x_c, y_c)$ be the coordinates of point P on the parabola. The slope of the line of y = -x is -1, so the slope of PA is 1 and the equation of PA has the form



$$y = x + b$$

Since P($x_c$, $y_c$) lies on this line, then

$$y_c = x_c + b$$

and $$b = y_c - x_c$$

The equation of PA is then

$y = x + (y_c - x_c)$ when (x, y) are the coordinates of any point on PA.

Solving the system $\begin{cases} y = -x \\ y = x + (y_c - x_c) \end{cases}$

determines the coordinates of A as $\left(\frac{x_c - y_c}{2}, \frac{y_c - x_c}{2}\right)$. Applying the distance formula,

$$\overline{PA} = d = \sqrt{\left[x_c - \frac{(x_c - y_c)}{2}\right]^2 + \left[y_c - \frac{(y_c - x_c)}{2}\right]^2}$$

$$= \sqrt{\left(\frac{x_c + y_c}{2}\right)^2 + \left(\frac{x_c + y_c}{2}\right)^2}$$

$$= \sqrt{\frac{(x_c + y_c)^2}{2}}$$

$$= \frac{x_c + y_c}{\sqrt{2}}$$

The equation of the parabola then follows as in previous solution.

(sketch)

6. a-d. (Sketch of each)

Exercises 6-3. Comments.

The standard equation form $(x-a)^2 + (y-b)^2 = r^2$ can be developed for circle with center at (a,b) and radius of r units so that specific equations in general form $x^2 + y^2 + Dx + Ey + F = 0$ can be transformed to the standard form in order to determine coordinates of the center and the radius value as in Exercise 6-3, Ex. 2. The converse statement should be discussed in order to complete the locus discussion.

The flashlight discussion can be actually shown in a darkened room to relate the three types of curve.

The locus definition of the ellipse can be shown effectively with the thumb tack method and the resulting ellipse used to locate the various parts of the ellipse.

Example 1 (or a similar type problem) should be carried through carefully in a class discussion and both the general and standard forms pointed out, with the general form used as the simplified result and the standard equation being the one from which the intercepts can be readily obtained by inspection.

A short method for plotting can be developed by use of the intercepts and the coordinates of the end points of the focal chords as developed in Exercise 6-3, Ex. 9. The axis, origin and symmetry of the ellipse can be exploited in forming coordinates of points on the ellipse for sketching. In Exercise 6-3, Ex. 10 utilizes these symmetry types.

The case of ellipses with centers at the origin and foci on the y-axis can be compared to similar types having foci on the x-axis to show the interchange of $a^2$ and $b^2$ values.

In Exercise 6-3, Ex. 4i needs completing the square in x and y to generate a standard equation form from which the sketch can be made. In all of Ex. 4, sketches should be based on properties of the ellipse, rather than plotting a set of selected coordinates. Parts j and k of Ex. 4 review sketching of the circle and parabola respectively. Since Ex. 5-7 involve useful properties of the ellipse, it may be well to discuss them rather fully in class.

**Exercises 6-3. Answers.**

1. $x^2 + y^2 - 4x + 2y - 4 = 0$
2. $(2, -3)$
3. a. $\frac{x^2}{25} + \frac{y^2}{9} = 1$; $9x^2 + 25y^2 = 225$.
   b. $\frac{x^2}{25} + \frac{y^2}{16} = 1$; $16x^2 + 25y^2 = 400$
   c. $\frac{x^2}{9} + \frac{y^2}{25} = 1$; $25x^2 + 9y^2 = 225$
   d. $\frac{x^2}{16} + \frac{y^2}{25} = 1$; $25x^2 + 16y^2 = 400$
   e. $\frac{x^2}{12} + \frac{y^2}{16} = 1$; $4x^2 + 3y^2 = 48$
4. a. [Sketch, ellipse with intercepts $(\pm 2,0)$ and $(0, \pm\sqrt{3})$]
   b. [ " " " " $(\pm 4,0)$ and $(0, \pm 3)$]
   c. [ " " " " $(\pm\sqrt{3},0)$ and $(0, \pm 2)$]
   d. [ " " " " $(\pm 3,0)$ and $(0, \pm 5)$]
   e. [ " " " " $(\pm 2,0)$ and $(0, \pm 1)$]
   f. [ " " " " $(\pm 2,0)$ and $(0, \pm 3)$]
   g. center at $(3,5)$, major axis ends at $(3,10)$ and $(3,0)$, and minor axis

ends at (0,5) and (6,5)]

h. [Sketch, ellipse with center at (-2,1), major axis ends at (-6,1) and (2,1), and minor axis ends at (-2,4) and (-2,-2)]

i. [Sketch, ellipse with center at (4,-5), major axis ends at (4,0) and (4,-10), and minor axis ends at (0,-5) and (8,-5)]

j. [Sketch, circle with center at (3,-2) and radius of 9]

k. [Sketch, parabola with vertex at (2,-1) and on points of (0,-2) and (4,-2)]

l. [Sketch, ellipse with center at (-1,2), major axis ends at (-4,2) and (2,2), and minor axis ends at (-1,4) and (-1,0)]

6. If P is not on the major axis, then the triangle inequality of the sum of 2 sides greater than the third gives $\overline{PF} + \overline{PF'} > 2c$ using $\triangle FPF'$. If P is either end of the major axis, then either $\overline{PF}$ or $\overline{PF'}$ will be greater than $\overline{FF'}$, so $\overline{PF} + \overline{PF'} > \overline{FF'}$. Since $\overline{PF} + \overline{PF'} = 2a$ and $\overline{FF'} = 2c$ from Ex. 5, then from the inequality $2a > 2c$ and $a > c$. If $a > c$, then $a^2 > c^2$ since a and c are positive quantities and $a^2 - c^2 > 0$. For all real values of b, $b^2 > 0$ so we can let $b^2 = a^2 - c^2$ and the ellipse equation becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The x-intercepts of the curve are $(\pm a,0)$ and the y-intercepts $(0, \pm b)$, so these coordinates represent the end points of the major and minor axes respectively of the ellipse.

8. Using $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the standard equation for ellipses having foci on the x-axis and center at (0,0), the (0,b) minor axis end point, and the focus of (-c,0), the distance between them is $\sqrt{(0+c)^2 + (b-0)^2} = \sqrt{b^2 + c^2}$. Since $b^2 = a^2 - c^2$, then $b^2 + c^2 = a^2$ and the distance $\sqrt{b^2 + c^2} = a$ which is one half the major axis length of 2a. Similar arguments hold in the other 3 cases as well as for the four cases using $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, the standard equation for ellipses having foci on the y-axis and center at (0,0).

9. Using F(c,0), the coordinates of one end of the focal chord at F will be $(c, \bar{y})$ when y is the ordinate for that point. Since $(c, \bar{y})$ lies on the ellipse, then the coordinates satisfy the equation of the ellipse and $\frac{c^2}{a^2} + \frac{\bar{y}^2}{b^2} = 1$. Then $\bar{y}^2 = \frac{a^2b^2 - b^2c^2}{a^2} = \frac{b^2(a^2 - c^2)}{a^2} = \frac{b^4}{a^2}$. Hence,

$\bar{y} = \pm \frac{b^2}{a}$ so the coordinates of the end points of the focal chord are $(c, \pm \frac{b^2}{a})$. Using the other focus F'(-c,0) in a similar manner determines the coordinates of its end points as $(-c, \pm \frac{b^2}{a})$.

*10. Let $(\bar{x}, \bar{y})$ be coordinates of one of the vertices of the square. Since ellipse is symmetric with respect to both axes and the origin, then $2\bar{x} = 2\bar{y}$ = length of side of the square. Hence $\bar{x}^2 = \bar{y}^2$ and from the equation of the ellipse $5\bar{x}^2 = 80$ or $\bar{x} = \pm 4 = \bar{y}$. Required coordinates are $(\pm 4, \pm 4)$.

Exercises 6-4. Comments.

A direct construction of the hyperbola by use of compasses can be used to emphasize the locus definition. If the foci are placed on an arbitrary line segment selected for $|d(P,F) - d(P,F')|$, then an arc of any radius drawn from either focus will intersect a second arc from the other focus, using the radius of the first arc increased by the length of $|d(P,F) - d(P,F')|$, in two points of the hyperbola. The two branches of the hyperbola can be determined by a few such point pairs and the various parts of the hyperbola identified. This construction also can be used to point out the symmetry of the curve for both axes and the origin.

It may be well to trace through in detail the algebraic manipulation of Example 1 which will be needed in Ex. 1 of the Exercises which involve hyperbolas and foci on the y-axis as well as the x-axis. Exercises 2-5 amplify text remarks and can be used to emphasize symmetry, coordinates of the vertices, and comparison of the hyperbola with the ellipse. The equations of hyperbolas with foci on the x-axis and center at the origin should be compared with those having foci on the y-axis for both the general and standard equation forms. Also it may be well to point out the difference in the a, b, and c relations for the ellipse and hyperbola.

In sketching the curves of Exercise 6, (note that part k is the equation of an ellipse) use can be made of the standard equation form to determine intercepts and symmetric point pairs. The rectangular hyperbola should be related to those with foci on the axes in a rotated form. Parts g and h can be related to hyperbolas by squaring both members of each equation, but point out that this introduces extraneous points into the locus. Parts i-k review completing the square method for equations involving quadratic and first degree terms, while part l can be related to a rectangular hyperbola by factoring.

The starred problem of Ex. 7 amplifies a text remark and the length of focal chord developed in Ex. 8 can be combined with the coordinates of the foci and vertices, as a short sketching method. The usual standard form of the hyperbola with center at (h, k) is developed in Ex. 9.

Exercises 6-4. Answers.

1. a. $7x^2 - 9y^2 = 63$; $\frac{x^2}{9} - \frac{y^2}{7} = 1$

   b. $8x^2 - y^2 = 8$; $\frac{x^2}{1} - \frac{y^2}{8} = 1$

   c. $7y^2 - 9x^2 = 63$; $\frac{y^2}{9} - \frac{x^2}{7} = 1$

   d. $8y^2 - x^2 = 8$; $\frac{y^2}{1} - \frac{x^2}{8} = 1$

   e. $3x^2 - y^2 = 3$; $\frac{x^2}{1} - \frac{y^2}{3} = 1$

3. Since $c > 0$, from the text (see 6-4c), the argument follows the same pattern as that used in Exercise 6-3 for the ellipse.

4. This argument follows the same pattern as that used in Exercise 6-3 for the ellipse.

5. Since vertices of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $(\pm a, 0)$, the transverse axis length is 2a, the value of the distant difference used to form the equation.

6. a. [Sketch of hyperbola with vertices at $(\pm 4, 0)$ and asymptotes of $y = \pm \frac{5}{4} x$. Do not show asymptotes.]

   b. [Sketch of hyperbola with vertices at $(\pm 5, 0)$ and asymptotes of $y = \pm \frac{4}{5} x$. Do not show asymptotes.]

   c. [Sketch of hyperbola with vertices at $(\pm 6, 0)$ and asymptotes of $y = x$. Do not show asymptotes.]

   d. [Sketch of hyperbola with vertices at $(0, \pm 6)$ and asymptotes of $y = x$. Do not show asymptotes.]

   e. [Sketch of rectangular hyperbola with points

| | | | | |
|---|---|---|---|---|
| x | 1 | 4 | -1 | -4 |
| y | 4 | 1 | -4 | -1 |

]

   f. [Sketch of rectangular hyperbola with points

| | | | | |
|---|---|---|---|---|
| x | 1 | 4 | -1 | -4 |
| y | -4 | -1 | 4 | 1 |

]

   g. [Sketch of the upper branch of hyperbola with vertex at $(0, 6)$ and asymptotes of $y = \pm x$. Do not show asymptotes.]

   h. [Sketch of the left branch of the hyperbola with vertex at $(-6, 0)$ and asymptotes at $y = \pm x$. Do not show asymptotes.]

   i. Complete the square in x and y to form $(x - 2)^2 - (y - 3)^2 = 1$. This shows the curve to be a hyperbola with center at $(2, 3)$, $a^2 = b^2 = 1$. [Sketch of hyperbola with vertices at $(1, 3)$ and $(3, 3)$, and asymptotes of

$y = x + 1$ and $y = -x + 5$. Do not show asymptotes.]

j. Complete the square in x and y to form $9(x-4)^2 - 16(y-3)^2 = 144$. This determines a hyperbola with center at (4,3) and $a^2 = 16$. [Sketch of hyperbola with vertices at (0,3) and (8,3), and asymptotes of $3x + 4y = 24$ and $3x - 4y = 0$. Do not show asymptotes.]

k. Complete the square to form $9(x-4)^2 + 16(y-3)^2 = 144$. This determines an ellipse with center at (4,3), major axis ends at (0,3) and (8,3), and minor axis ends at (4,0) and (4,6). [Sketch of ellipse.]

l. Factor to form $y(x-2) - 1 = 0$, or $(x-2)y = 1$. This determines a rectangular hyperbola with center at (2,0). [Sketch of rectangular hyperbola with points , asymptotes of $x = 2$ and $y = 0$. Do not show asymptotes.]

| x | $\frac{5}{2}$ | 3 | 4 | $\frac{3}{2}$ | 1 | 0 |
|---|---|---|---|---|---|---|
| y | 2 | 1 | $\frac{1}{2}$ | -2 | -1 | $-\frac{1}{2}$ |

*11. From the definition of the hyperbola,

$$\left| \sqrt{(x-a)^2 + (y-a)^2} - \sqrt{(x+a)^2 + (y+a)^2} \right| = 2a$$

Simplifying this equality gives the equation as $xy = \frac{a^2}{2}$ or $xy = k$, $k > 0$ since $\frac{a^2}{2} > 0$ for all real a. If $F(a, -a)$ and $F'(-a, a)$ are used with the distance difference, the equation becomes $xy = \frac{-a^2}{2}$ or $xy = k$, $k < 0$. The graph of $xy = 0$ is the set of points lying on the coordinate axes.

*12. The coordinates are $(c, \pm\frac{b^2}{a})$. The development follows the same pattern as for the ellipse. The length of the focal chord is $\frac{2b^2}{a}$. It applies also to the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Exercises 6-5. Comments.

Note three forms used to express asymptotes. The use of these forms should be varied since the exercises make use of them.

In Exercises 6-5, Ex. 1 follows the example through part f. The coordinates of the vertices can be easily found from the standard equation form and computation of one other coordinate pair will suffice for the sketch if the symmetric point and asymptotes are used. Use can also be made of the length of the focal chords developed in Ex. 8 of Exercises 6-4. Parts h and i will need three or more coordinate pairs together with symmetric points and asymptotes. Parts i and j are a bit more difficult, but review use of completing the square and factoring to produce a general form from which information about the hyperbolas can be gained. This is to be preferred as against the computation of many coordinate pairs to be plotted.

Exercises 2 and 3 are general and will demand an understanding of the special case arguments of the text. In Ex. 4-9 the various expression forms for the asymptotes are used and sketches are to be made only for those having vertices as integer coordinates. Exercise 10 generalizes the methods used in Ex. 4-9.

Exercises 6-5. Answers.

1. a. [Sketch, hyperbola with asymptotes $y = \pm \frac{3}{2}x$, and vertices $(\pm 2,0)$]

b. [Sketch, hyperbola with asymptotes $y = \pm \frac{1}{2}x$, and vertices $(\pm 1,0)$]

c. [Sketch, hyperbola with asymptotes $y = \pm \frac{3}{2}x$, and vertices $(\pm 3,0)$]

d. [Sketch, hyperbola with asymptotes $y = \pm \frac{1}{2}x$, and vertices $(\pm 2,0)$]

e. [Sketch, hyperbola with asymptotes $y = \pm 3x$, and vertices $(\pm 1,0)$]

f. [Sketch, hyperbola with asymptotes $y = \pm 3x$, and vertices $(0, \pm 3)$]

g. [Sketch, hyperbola with asymptotes $x = 0$ and $y = 0$, selected points

| x | 1 | 4 | -1 | -4 |
|---|---|---|---|---|
| y | 4 | 1 | -4 | -1 |

]

h. [Sketch, hyperbola with asymptotes $x = 0$ and $y = 0$, selected points

| x | 1 | 4 | -1 | -4 |
|---|---|---|---|---|
| y | -4 | -1 | 4 | 1 |

]

i. Completing the square in x and y and factoring forms $25(x-2)^2 - 4(y-5)^2 = 100$. Asymptotes are given by $25(x-2)^2 - 4(y-5)^2 = 0$, or $[5(x-2) - 2(y-5)]\,[5(x-2) + 2(y-5)] = 0$. This gives asymptotes of $5(x-2) - 2(y-5) = 0$ or $5x - 2y = 0$, and $5(x-2) + 2(y-5) = 0$ or $5x + 2y = 20$. [Sketch, hyperbola with asymptotes $y = \frac{5}{2}x$ and $y = -\frac{5}{2}x + 10$, and vertices at $(4,5)$ and $(0,5)$.]

j. Factoring the equation to form $(x-3)(y+2) = 1$ gives equation of a rectangular hyperbola with center at $(3, -2)$. The asymptotes follow from the form

$(x-3)(y+2) = 0$ as $x = 3$ and $y = -2$. [Sketch, hyperbola with center at $(3,-2)$, asymptotes $x = 3$ and $y = -2$, selected points

| x | $\frac{7}{2}$ | 4 | 5 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| y | 0 | -1 | $-\frac{3}{2}$ | $-\frac{7}{3}$ | $-\frac{5}{2}$ | -3 |

]

2. From problem conditions, $\sqrt{\frac{2}{5}}\,(x-\sqrt{x^2-5}) = 0.0001$. Solving for x,

$$\sqrt{\frac{2}{5}}x - 0.0001 = \sqrt{\frac{2}{5}}\ \sqrt{x^2-5}$$

$$\frac{2}{5}x^2 - 2\ \sqrt{\frac{2}{5}}\,(0.0001)x + 0.0001^2 = \frac{2}{5}\,(x^2-5)$$

$$x = \frac{2 + 0.0001^2}{2\sqrt{\frac{2}{5}}\,(0.0001)} \approx \frac{2}{0.00004\ \sqrt{10}} \approx 15.8(10^3)$$

3. Form $b^2x^2 - a^2y^2 = a^2b^2$, then $b^2x^2 - a^2y^2 = 0$ is equation of asymptotes or $y = \pm\frac{b}{a}x$. The proof follows the pattern of the text example to form

$$y_1 - y_2 = \frac{ab}{x + \sqrt{x^2-a^2}}$$

If x becomes large, say $x \geq N$, then $x + \sqrt{x^2-a^2}$ becomes large (actually $x + \sqrt{x^2-a^2} \geq N$), and $\frac{1}{x + \sqrt{x^2-a^2}}$ becomes small (actually, $0 \leq \frac{1}{x+\sqrt{x^2-a^2}} \leq \frac{1}{N}$). From this it follows that $y_1 - y_2$ becomes smaller (actually, $0 \leq y_1 - y_2 \leq \frac{ab}{x + \sqrt{x^2-a^2}} \leq \frac{ab}{N}$).

4. Since asymptotes given by $3x - 5y = 0$ and $3x + 5y = 0$, then they

are also given by $(3x - 5y)(3x + 5y) = 0$, or $9x^2 - 25y^2 = 0$. Reversing the procedure used to find the asymptote equations gives the hyperbola equation as $9x^2 - 25y^2 = k$, where k is the right member constant. But the hyperbola passes through the point of (2,3) so by use of the locus property $9(2)^2 - 25(3)^2 = k$ and k is found to be -189. Hence the required equation is $9x^2 - 25y^2 = -189$.

5. $9x^2 - 25y^2 = 200$.
6. $16x^2 - y^2 = -4$. [Sketch; hyperbola with center (0,0), asymptotes $y = \pm 4x$, vertices $(0, \pm 2)$].
7. $16x^2 - y^2 = 64$. [Sketch; hyperbola with center (0,0), asymptotes $y = \pm 4x$, vertices $(\pm 2, 0)$].
8. $3x^2 - 5y^2 = -33$.
9. $2x^2 - 7y^2 = 18$.
10. $a^2x^2 - b^2y^2 = a^2b^2$
11. Sketches of hyperbolas as indicated. Each has asymptotes $y = \pm x$ and vertices at a. $(\pm 4,0)$. b. $(\pm 3,0)$. c. $(\pm 2,0)$ d. $(\pm 1,0)$ e. $(0, \pm 1)$ f. $(0, \pm 2)$ g. The curve consists of the two lines of the asymptotes. h. $(0, \pm 4)$

Exercises 6-6. Comments.

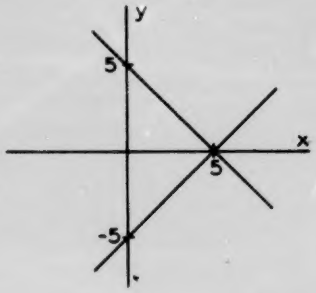
A review of the meaning of the solution set of an equation or inequality and their graphs will be helpful in establishing the meaning of equivalent systems. The use of a single equation for a two-line locus was used previously with hyperbola asymptotes and will be used again in Section 6-9 with quadratic systems.

The solution set for an inequality can be determined from the property that the curve of the related equation partitions the x-y plane into disjoint point sets. A text of point coordinates, such as (0,0), will quickly determine the solution set of the inequality.

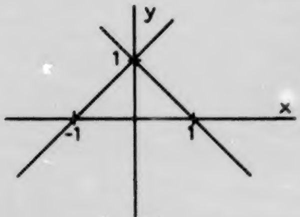
In Exercises 6-6, Ex. 1 provides problems of Example 2 type, with the starred parts having squared terms concealed. The conic section curves are reviewed by Ex. 2, with the starred parts treating the more general cases.

Exercises 6-6. Answers.

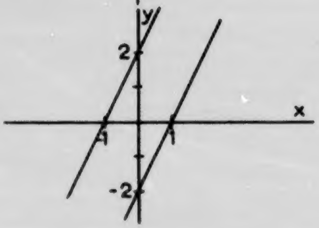
1. a.



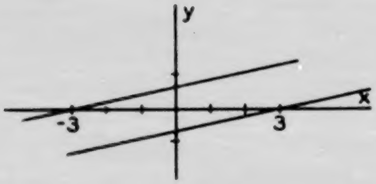
b.



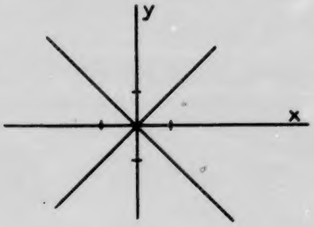
c.



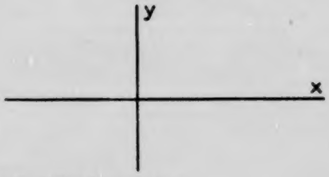
d.



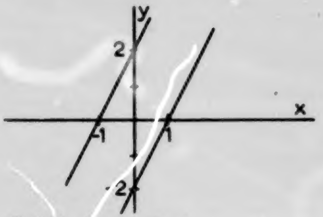
e.



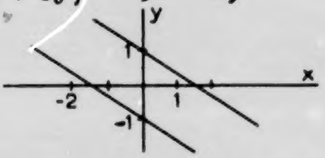
f.



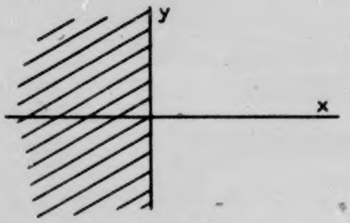
g. Simplifying to $(x+y)^2 - 4 = 0$



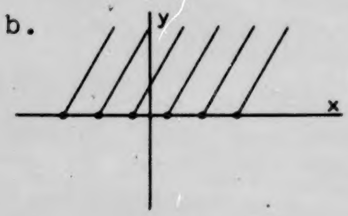
h. Simplifying to $(2x+3y)^2 - 9 = 0$,



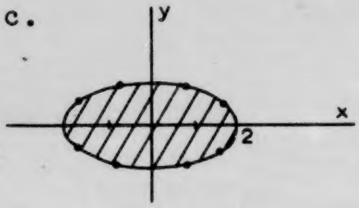
2. a.



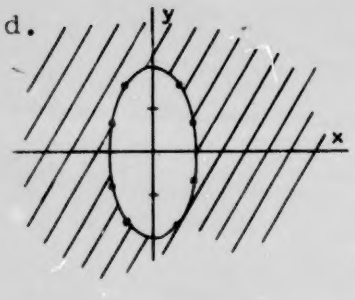
2. b.



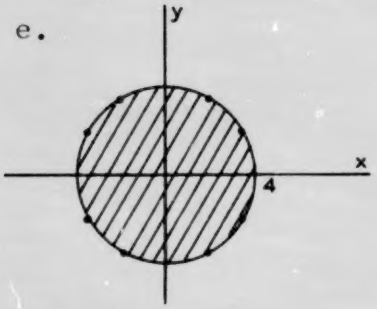
2. c.



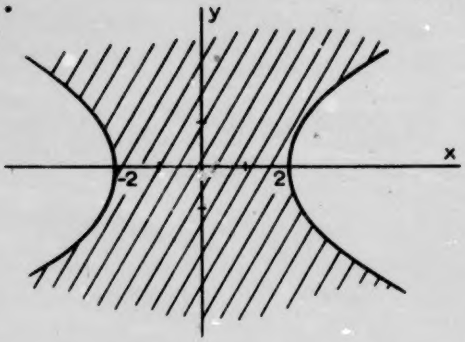
2. d.



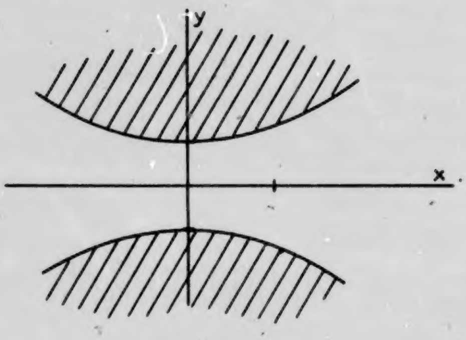
2. e.



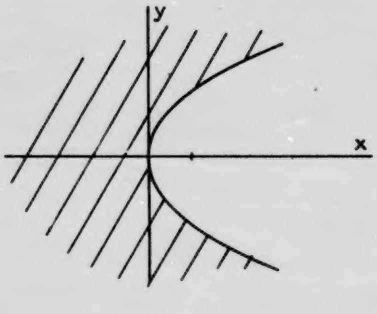
2. f.



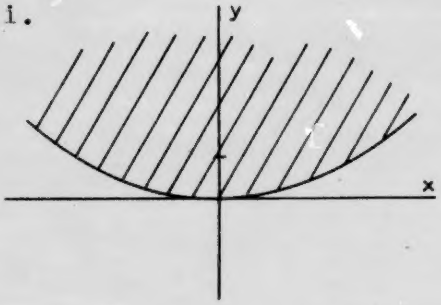
2. g.



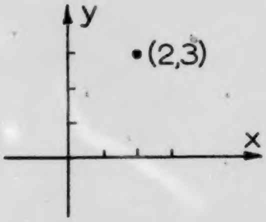
2. h.



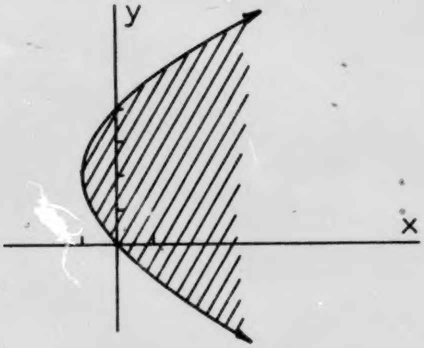
2. i.



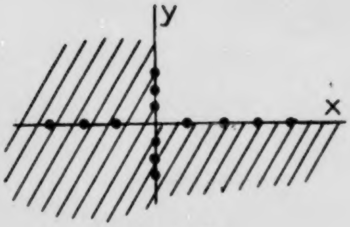
*j. Simplify to $(x-2)^2 + (y-3)^2 \leq 0$. The only element of the solution set is (2,3), so the graph consists of this single point.



*k. Simplify to $(y-2)^2 < 4(x+1)$. The graph of $(y-2)^2 = 4(x+1)$ is a parabola with vertex at (-1,2) opening to the right



*l.



Exercises 6-7. Comments.

Discuss solution set of an equation in $x$ and $y$ both in terms of a set of ordered number pairs and the graphic representation involving real number pairs using specific examples leading into the solution set of a system as the intersection of the solution sets of the component equations or the graphic intersection of their curves. Note that the solution set is empty, not a case of no solution, when the system is inconsistent.

Exercises 6-7. Answers.

1. Yes. [Sketch of $\left\{\begin{matrix} 2x + 3y = 4 \\ 8x - 7y = 16 \end{matrix}\right.$]. $(2,0)$ is point of intersection for lines of $E_1$ and $E_2$.
2. Yes. [Sketch of $\left\{\begin{matrix} x + y = 5 \\ x - y = 3 \end{matrix}\right.$]. $(4,1)$ is point of intersection for lines of $E_1$ and $E_2$.
3. Yes. Graphs of $E_1$ and $E_2$ are same line. This indicates dependency of the equations.
4. No. [Sketch of $\left\{\begin{matrix} x + 4y = 13 \\ 2x + 8y = 14 \end{matrix}\right.$]. Lines of graph are parallel indicating the system is inconsistent and has the solution set $\emptyset$.

*5. By the Distributive Law, $2x + 8y - 14 = 2(x + 4y - 7)$, so if $2x + 8y - 14 = 0$, then $x + 4y - 7 = 0$ and $x + 4y = 7$. But the loci of $x + 4y = 7$ and $x + 4y = 13$ are parallel lines so their solution sets have $\emptyset$ as an intersection.

6. Single element set, empty set, or infinite set. Lines of the components intersect, are parallel, are the same line respectively.
7. a. $b \neq 44$.

b. No value of b will cause the solution set to have a single element since the system is dependent for $b = 44$, and inconsistent for all other values of b.

c. $b = 44$.

8. Yes. Curves are parabolas symmetric to the y-axis, so a second element of the solution set would be $(-1,1)$. [Sketch $\begin{cases} y = x^2 \\ y = 2 - x^2 \end{cases}$; indicate points $(1, 1)$ and $(-1,1)$].

9. Yes. [Sketch $\begin{cases} y = 2x^2 \\ y = 4 - 2x^2 \end{cases}$]. Since the curves are parabolas symmetric to the y-axis, a second element of the solution set is $(-1,2)$.

10. Yes. [Sketch of $\begin{matrix} 3x + 2y = 7 \\ 5x - y = 3 \\ 16x + 2y = 20 \end{matrix}$ intersecting at $(1,2)$].

11. The graphs of the components of the system must intersect at a common point.

*12. If S has a single element in its solution set, then the lines of the components would have a common point of intersection so that $ax + by - c = 0$ and $dx + ey - f = 0$ would be satisfied by the point of intersection. Hence, any constants m and n, not both zero, cause $m(ax + by - c) + n(dx + ey - f) = 0$ to be satisfied by this point of intersection. This means that $m(ax + by - c) + n(dx + ey - f) = 0$ will be the equation of a line through the intersection of the lines of the system. This is called a family or pencil of lines and is a modified form of linear combination that will be used in the next section.

## Exercises 6-8 Comments.

Use examples showing operations that produce equivalent equations as well as those that do not. Review meaning of the solution set of a system and equivalent systems before stating the definitions. Give examples showing that equivalence of systems cannot be always determined by inspection of components of the systems. Relate equivalence of systems to intersections of the graphic forms. Stress importance of Principle 6-1 as the general attack upon finding the solution set for a system of equations, with Principle 6-2 coming out of the example.

The definition of linear combination used here is somewhat more restricted than the definition sometimes used in more advanced courses, in which either a or b or both may be zero. The linear systems of the examples are carried out in detail to clarify the logical basis of the usual combination or elimination methods actually used in solving systems of equations.

## Exercises 6-8. Answers.

1. Yes; since (3,5) satisfies both systems
2. Yes; since (6,2) satisfies both systems.
3. Yes; by the replacement principle since $x + y = 10$ is equivalent to $2x + 2y = 20$.
4. No; the points of intersection are different.
5. Yes [Sketch of $S_1$ and $S_2$]. $S_3 \begin{cases} E_1 & x - y + 5 = 0 \\ E_2 & x + y = 5 \end{cases}$ is another system equivalent to $S_1$ and $S_2$ since

$$(x^2 - y^2) - 16 = [(x - y) - 4][(x - y) + 4].$$

6. Yes. [Sketch of $S_1$ and $S_2$]. $S_3 \begin{cases} E_1 \; x - y + 4 = 0 \\ E_2 \; x^2 + 4y^2 = 4 \end{cases}$ is another system equivalent to $S_1$ and $S_2$.

7. S and S' are not equivalent since $x^2 + y^2 = 16$ is not equivalent to $x^2 - y^2 = 9$, or graphs will show different intersection points.

8. $S_1$ and $S_2$ are not equivalent since neither factor of $x^2 + y^2 - 16$ is equivalent to $x + y - 7$, or graphs will show different intersection points.

9. $S_1 \begin{cases} a(x+y-1) + b(2x-y-4) = 0 \\ 2x - y = 4 \end{cases}$, $S_2 \begin{cases} a(x+y-1) + b(2x-y-4) = 0 \\ x + y = 1 \end{cases}$

10. $S_1 \begin{cases} a(x^2+x-2y+4) + b(x+y-1) = 0 \\ x^2 + y - 1 = 0 \end{cases}$. Let a = 1 and b = 5, then $S_2 \begin{cases} x - 3y + 5 = 0 \\ x + y - 1 = 0 \end{cases}$.

11. $S_1 \begin{cases} a(7x+5y-11) + b(3x-2y-13) = 0 \\ 3x - 2y = 13 \end{cases}$. Let a = 2 and b = 5, then $S_2 \begin{cases} 29x - 87 = 0 \\ 3x - 2y = 13 \end{cases}$ and $S_3 \begin{cases} x = 3 \\ 3x - 2y = 13. \end{cases}$ Using x = 3 in second equation, $S_4 \; \{(3, -2)\}$

12. $\{(5,6)\}$

*13. Proof follows directly from the definition of equivalent systems.

*14. Proof follows same pattern used in the text to prove $S_1$ equivalent to $S_2$.

*15. Suppose $(x_1, y_1)$ is a pair of values satisfying $S_1$, so that $f(x_1, y_1) = 0$, $g(x_1, y_1) = 0$. Then it will follow that $af(x_1, y_1) + bg(x_1, y_1) = 0$ and $cf(x_1, y_1) + dg(x_1, y_1) = 0$, so that each solution of $S_1$ is a solution of $S_2$. Conversely, suppose that $ad - bc \neq 0$, and that $(x_1, y_1)$ is a pair satisfying $S_2$. Then

$$af(x_1, y_1) + bg(x_1, y_1) = 0$$

$$cf(x_1, y_1) + dg(x_1, y_1) = 0.$$

Thus $d(af(x_1, y_1) + bg(x_1, y_1)) - b(cf(x_1, y_1) + dg(x_1, y_1)) = 0$, which reduces to $(ad - bc)f(x_1, y_1) = 0$. Since $ad - bc \neq 0$ it follows that $f(x_1, y_1) = 0$. Similarly we may show that $g(x_1, y_1) = 0$. Hence each solution of $S_2$ is a solution of $S_1$, and we conclude that $S_1$ and $S_2$ are equivalent.

Exercises 6-9. Comments.

In developing an equivalent system by the method of linear combination it may be helpful to write the components of the system vertically to determine a and b values. That is, for $S\begin{cases} E_1 & x^2 + 2y - 1 = 0 \\ E_2 & 2x - 3y + 2 = 0 \end{cases}$ writing the components as $\left| \begin{array}{l} x^2 + 2y - 1 = 0 \\ 2x - 3y + 2 = 0 \end{array} \right.$ the values for a and b can perhaps be more readily seen. This is the familiar method given in most algebra texts, but it is important to actually write the equivalent system as $S_1\begin{cases} E_3 & 3x^2 + 4x + 1 = 0 \\ E_2 & 2x - 3y + 2 = 0 \end{cases}$, using $a = 3$ and $b = 2$, in order to insure that proper use is made of the final substitution procedure to produce elements of the solution set.

Exercises 6-9, Ex. 1-2 give practice in choice of a and b values with linear combinations which may or may not be needed, depending upon student abilities. The method of using equivalent systems can be done in detail in Ex. 4 with part a using the union method and the rest by linear combination. Part b of Ex. 4 shows the case of tangency, and part d the case of no intersection for the curves of the component equations of the systems. In Ex. 5-6 perhaps the strict sequence of written equivalent systems can be abbreviated or even dropped, but the principle of the equivalent systems should be followed through the solution. Part b of Ex. 5 involves the tangent relation, with part c having no intersection for the curves of the component equations of the system. Part d of Ex. 5 is more difficult, particularly with respect to the curves. Parts h-j use the method of Example 5, elimination of the constant when an xy-term is present in the quadratic system. This is a general method for systems of this type, but is best used when the resulting equation is readily factorable into linear factors. Parts k and l involve use of substitutions since first degree terms are present.

Exercises 6-9. Answers.

1. a. a = 1, b = -1.
   b. a = 3, b = 7.
   c. a = 3, b = 2.
   d. a = 2, b = 9.

(Note - each a and b value given have proportional values which may also be used, that is if a = 2 and b = 3, then other forms are a = 2k and b = 3k.)

2. a. a = 1, b = 1.
   b. a = 5, b = 1.
   c. a = 11, b = 5.
   d. a = -11, b = 5.

3. a. $a = -2$, $b = 1$ will remove x terms.

   b. $a = 3$, $b = 1$ " " " "

   c. $a = 5$, $b = 2$ " " y terms.

   d. $a = 1$, $b = -2$ " " " "

4. a. $S_1 \begin{cases} E_3 & (x-3)(x-1) = 0 \\ E_2 & x - y + 1 = 0 \end{cases}$ is equivalent to

   $$S_2 \begin{cases} E_4 & x - 3 = 0 \\ E_2 & x - y + 1 = 0 \end{cases} \cup \; S_3 \begin{cases} E_5 & x - 1 = 0 \\ E_2 & x - y + 1 = 0 \end{cases}$$

   using $x = 3$ with $E_2$ gives element (3, 4) and using $x = 1$ with $E_2$ gives element (1, 2). The solution set of S is $\{(3, 4), (1, 2)\}$.

   b. $S_1 \begin{cases} E_3 & a(x^2 - 5x + 3y + 1) + b(x + y - 5) = 0 \\ E_2 & x + y - 5 = 0 \end{cases}$

   or

   $S_2 \begin{cases} E_4 & x^2 - 8x + 16 = 0 \\ E_2 & x + y - 5 = 0 \end{cases}$ by use of $a = 1$, $b = -3$.

   $S_3 \begin{cases} E_5 & (x-4)(x-4) = 0 \\ E_2 & x + y - 5 = 0 \end{cases}$ is equivalent to

   $S_4 \begin{cases} E_6 & x - 4 = 0 \\ E_2 & x + y - 5 = 0 \end{cases} \cup \; S_5 \begin{cases} E_7 & x - 4 = 0 \\ E_2 & x + y - 5 = 0. \end{cases}$ Each of these has the same element of (4,1) as its solution set, so the union of these gives the solution set of S as $\{(4,1)\}$.

   c. The method of linear combination produces the equivalent system $S_1 \begin{cases} E_3 & x^2 + 4x + 3 = 0 \\ E_2 & x - y + 10 = 0 \end{cases}$. This reduces to the equivalent system $S_2 \begin{cases} E_4 & x + 3 = 0 \\ E_2 & x - y + 10 = 0 \end{cases}$

$$\cup \quad S_3 \begin{cases} E_5 & x+1=0 \\ E_2 & x-y+10=0 \end{cases} .$$

The solution set of S is $\{(-3,7), (-1,9)\}$.

d. By linear combination an equivalent system is

$$S_1 \begin{cases} E_3 & x^4-16=0 \\ E_2 & y=2x+7 \end{cases} \quad \text{or} \quad S_2 \begin{cases} E_4 & (x-2)(x+2)(x^2+4)=0 \\ E_2 & y=2x+7 \end{cases}$$

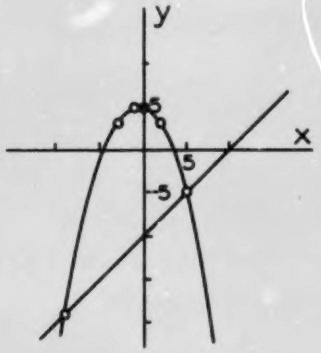
This is equivalent to

$$S_3 \begin{cases} E_5 & x-2=0 \\ E_2 & y=2x+7 \end{cases} \cup \; S_4 \begin{cases} E_6 & x+2=0 \\ E_2 & y=2x+7 \end{cases} \cup \; S_5 \begin{cases} E_7 & x^2+4=0 \\ E_2 & y=2x+7 \end{cases}$$

The union of these sets gives the solution set of S as $\{(2,11), (-2,3), (2i, 7+4i), (-2i, 7-4i)\}$

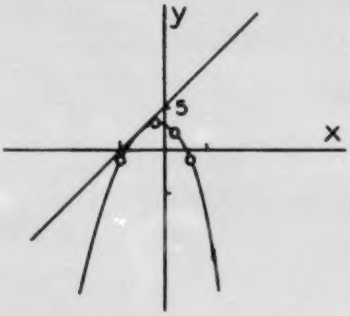
5. a. $(5, -5)$, $(-9, -19)$.

| x | 0 | 2 | 3 | -1 | -3 |
|---|---|---|---|---|---|
| y | 5 | 3 | 1 | 5 | 3 |

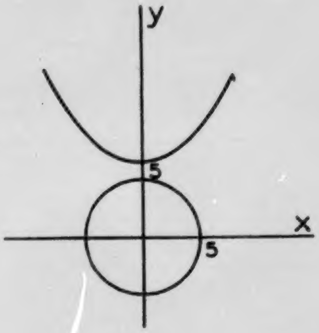


b. $(-3, 2)$

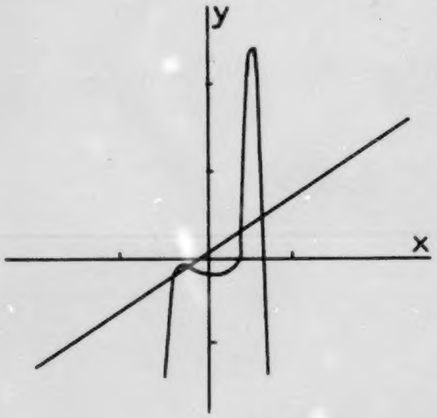
| x | 1 | 3 | -1 | -5 |
|---|---|---|---|---|
| y | 2 | -1 | 3 | -1 |



c. $(\pm i\sqrt{11}, -4)$ ; $(\pm 2i, 3)$



d. $(2, \frac{5}{3})$ ; $(-2,-1)$ ; $(1,1)$ ; $(-1,-\frac{1}{3})$



Line

$$\left[\begin{array}{c|cc} x & 7 & -2 \\ \hline y & 5 & -1 \end{array}\right]$$

Curve

$$\left[\begin{array}{c|ccc} x & 0 & 2 & 3 \\ \hline y & -1 & -\frac{5}{3} & 11 \end{array}\right]$$

By linear combination the equivalent system $S_1 \begin{cases} E_3 & x^4 - 5x^2 + 4 = 0 \\ E_2 & 2x - 3y + 1 = 0 \end{cases}$ is produced. This simplifies to the equivalent system

$$S_2\begin{cases} E_3 & x^2 - 4 = 0 \\ E_2 & 2x - 3y + 1 = 0 \end{cases} \cup \ S_3\begin{cases} E_4 & x^2 - 1 = 0 \\ E_2 & 2x - 3y + 1 = 0 \end{cases}$$

6. a. $(-6, -19)$ ; $(-3, -10)$.
   b. $(-3,-6)$ ; $(-2, -5)$
   c. $(-2,-5)$ ; $(-1, -3)$
   d. $(-\frac{1}{3}, 1 - \frac{\sqrt{2}}{3})$ ; $(1, 1 + \sqrt{2})$.
   e. $(1, 1)$

f. $(2, 9)$ ; $(-2, 11)$

g. $(5, 2)$ ; $(5, -2)$ ; $(-3, 21)$ ; $(-3, -21)$

h. Eliminate the constant to produce the linear combination equation of $12x^2 + 4xy - 5y^2 = 0$ which can be factored to form $(6x + 5y)(2x - y) = 0$. Then the solution set for S can be obtained from

$$S' \begin{cases} 4x^2 - 3y^2 + 2 = 0 \\ 6x + 5y = 0 \end{cases} \cup \; S'' \begin{cases} 4x^2 - 3y^2 + 2 = 0 \\ 2x - y = 0 \end{cases}$$

as $\{(\frac{1}{2}, 1), (-\frac{1}{2}, -1), (\frac{5}{2}, -3), (-\frac{5}{2}, 3)$

i. The constant term can be eliminated by linear combination to form $(x + y)^2 = 0$. Then the solution set for S can be obtained for

$$S' \begin{cases} x + y = 0 \\ y^2 + 2xy + 5x - 6 = 0 \end{cases} \cup \; S'' \begin{cases} x + y = 0 \\ y^2 + 2xy + 5x - 6 = 0 \end{cases}$$

as $\{(3, -3), (-2, 2)\}$.

j. Elimination of the constant term gives $2x^2 + 3xy - 2y^2 = 0$, so the solution set can be found for

$$S' \begin{cases} 2x - y = 0 \\ y^2 + 2xy + 5x - 6 = 0 \end{cases} \cup \; S'' \begin{cases} x + 2y = 0 \\ y^2 + 2xy + 5x - 6 = 0 \end{cases}$$

as $\{(1,2), (-1,-2), (4,-2), (-4,2)\}$

*k. Elimination of the constant gives the equivalent system

$$S_1 \begin{cases} E_1 \; y^2 - 2xy + y - 4 = 0 \\ E_3 \; y = x^2 \end{cases}$$

or

$$S_2 \begin{cases} E_4 \; x = \dfrac{y^2 - 4}{2y} \\ E_3 \; y = x^2 \end{cases}.$$

Substitution of $E_4$ into $E_3$ gives

$$S_3 \begin{cases} E_4 \; x = \dfrac{y^2 - 4}{2y} \\ E_3 \; y^2 = 4 \end{cases}.$$

From this the solution set of S is

$$\{(2,4), (-2,4), (2i, -4), (-2i, -4)\}.$$

*1. Noting that $x^2 - 2xy + y^2 + 4$ can be written as $(x - y)^2 = 4$, from the equivalent system

$$S_1 \begin{cases} E_1 \; 5x^2 + 8xy - 5y^2 + 6x + 4y - 2 = 0 \\ E_3 \; x - y - 2 = 0 \end{cases} \cup S_2 \begin{cases} E_1 \; 5x^2+8xy-5y^2+6x+4y-2=0 \\ E_3 \; x - y + 2 = 0 \end{cases}$$

The solution set of S is $\{(\frac{5}{4}, -\frac{3}{4}), (-3,-5), (-\frac{7}{4}, \frac{1}{4}), (1,3)\}$.

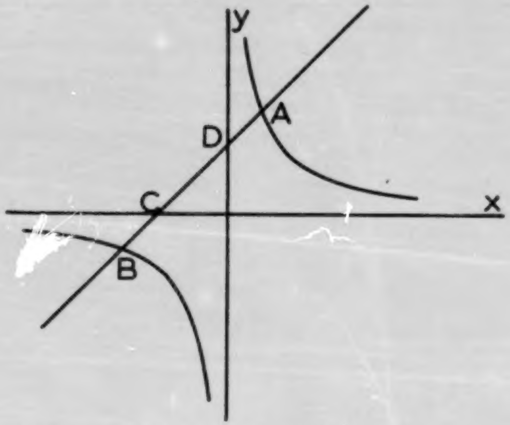
*7. If line is tangent to the circle, then the system $S \begin{cases} E_1 \; y = mx + b \\ E_2 \; x^2 + y^2 = a^2 \end{cases}$ must have a single element in its solution set. In the equivalent system

$$S_1 \begin{cases} E_1 \; y = mx + b \\ E_2 \; x^2 + (mx + b)^2 = a^2 \end{cases}$$

$E_2$ must have a single element in its solution set. Hence the discriminant for the quadratic $E_2$ must equal zero. From this condition it follows that $b = \pm a \sqrt{m^2 + 1}$

*8. Solve the system $S \begin{cases} E_1 \; xy = k, \; k > 0 \\ E_2 \; y = mx + b \end{cases}$ to obtain coordinates of points of intersection $A(\frac{-b + \sqrt{b^2 + 4k}}{2m}, \frac{b + \sqrt{b^2 + 4k}}{2}$ and $B(\frac{-b - \sqrt{b^2 + 4k}}{2m}, \frac{b - \sqrt{b^2 + 4k}}{2})$. Using $C(-\frac{b}{m}, 0)$

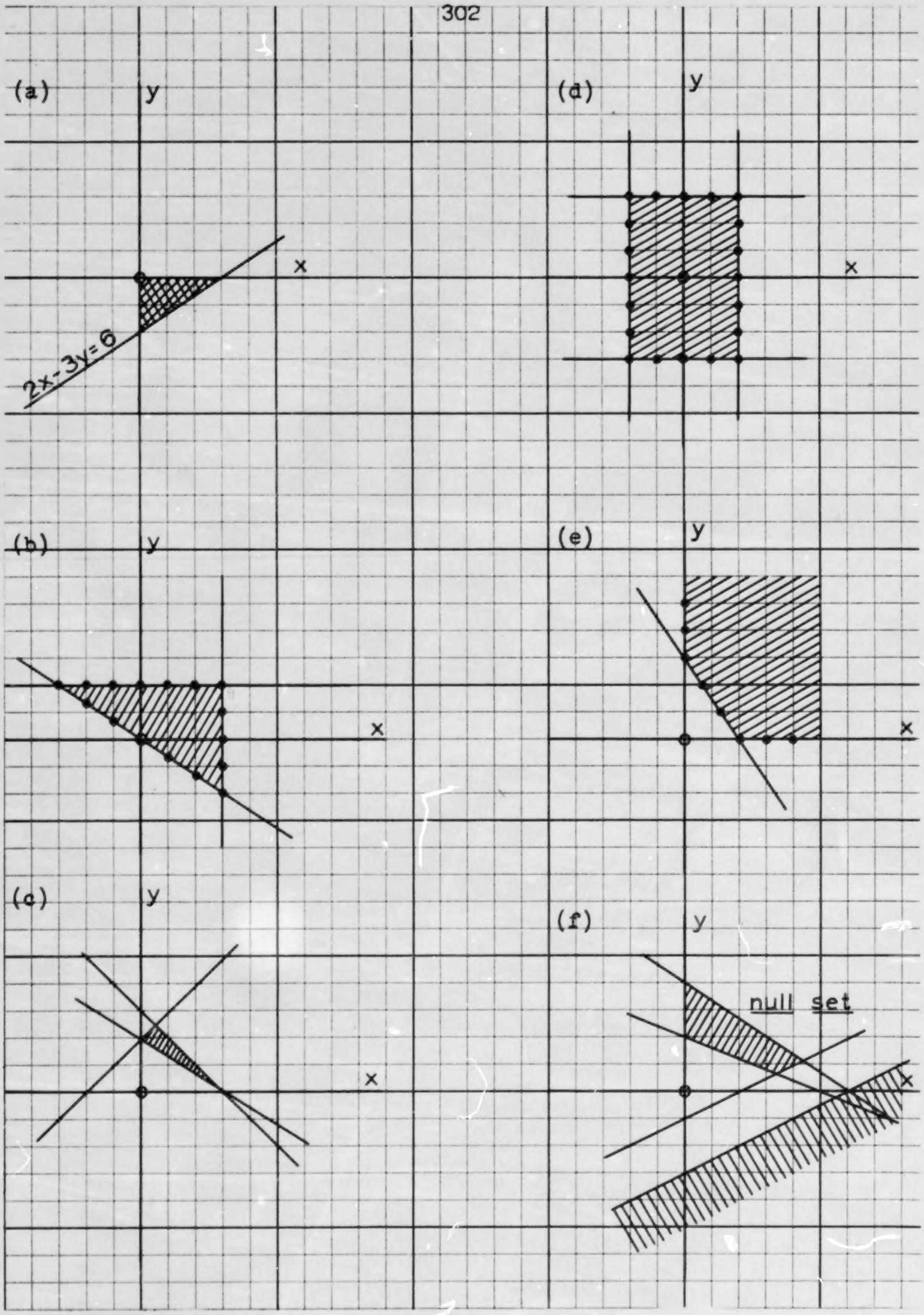
and D(0, b),



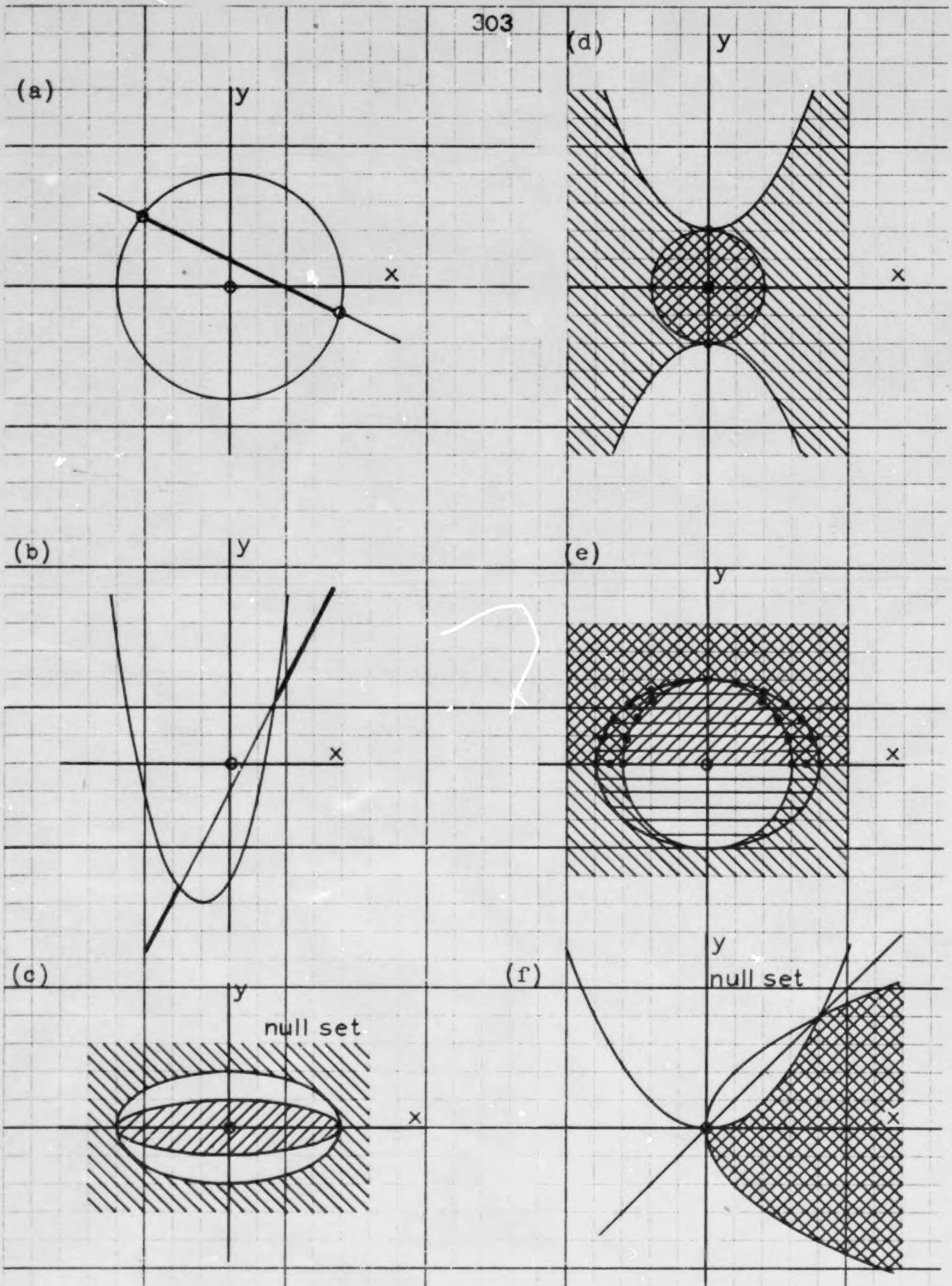
show that d(A,D) = d(B,C).

Exercises 6-10. Comments.

This section reviews previous graphic work with inequalities, both linear and quadratic. In Exercises 6-10, Ex. 1 involves linear systems of inequalities, with part f having an empty solution set. In Ex. 2 part c may be changed to involve $\geq$ and $\leq$ relations so that the solution set of the system would consist of the coordinates of the two intersection points. Similarly in part f, use of weak inequalities would give a solution set of two elements.


(a)
y
x
2x-3y=6
(d)
y
x
(b)
y
x
(e)
y
x
(c)
y
x
(f)
y
null set
x

303



## Teacher's Manual

## Chapter 7

### Introduction

Systems of first degree equations arise in many branches of mathematics and science as well as in many modern theories of economics and in business problems, particularly those concerned with inventory and production questions. The geometrical significance of the subject is emphasized by our presentation of the geometry along with the algebra of the problem.

With the current use of computing machines to solve engineering and scientific problems, this subject has become one of the most important branches of applied mathematics. Every day industrial and research organizations must solve systems of first degree equations, some of them with hundreds of equations and hundreds of variables. A thorough understanding of the simpler cases is therefore a necessity for any one hoping to take almost any kind of mathematical job in industry or scientific research.

The central problem studied in this chapter is an algebraic one: under what circumstances do two or more equations in three variables with real coefficients have common solutions, and if there are common solutions, how many are there and how are they related to one another? Because we restrict our attention to first degree equations with real coefficients having only three variables, we are able to translate the problem into geometric language. This translation makes it possible to cast our results in the form of statements about planes in three dimensional space in such a way that statements about common solutions of the equations become statements about configurations of planes and their intersections. The insights gained in this way are perhaps most strikingly illustrated by the diagrams in Figure 7 - 9b where the many types of intersection and parallelism of planes are used to describe the types of solution sets that may be expected when a system of three first degree equations in three variables is studied. In this presentation, p. 416 gives the cases where there are solutions, p. 417 gives the cases where the solution set is empty. Although it is not essential that the student understand all these cases, some students will enjoy the opportunity to see a classification of this kind. Thus those students who learn to handle the ideas presented in discussing the correspondence between the geometry and algebra of these (and other) systems of equations will benefit from the ability to visualize the geometry. If they go on to the study of mathematics at college, they will find that the development of this ability gives them real advantages.

### Degrees of Freedom

One of the basic ideas that will throw light on the approach to the problems studied in this chapter is the concept of degrees of freedom. An understanding of this concept will improve the teacher's intuitions about all the problems discussed here. (For a more sophisticated treatment of this question and of linear dependence, see Birkhoff-MacLane, Survey of Modern Algebra, p. 166ff.)

A point in space has 3 coordinates (x,y,z). It is said to have 3 degrees of freedom since each of the variables may be assigned arbitrary values. As each of x, y, and z assumes all possible real values, the point (x,y,z) assumes all possible positions in space. If, however, the values of the variables are constrained to satisfy a single equation (here an equation of first degree - the equation of a plane), the number of variables that may be assigned arbitrary values is reduced to 2. The point now has only 2 degrees of freedom and is constrained to remain in the plane whose equation is given. We say that the number triples in the solution set of the equation can be described in terms of 2 parameters. (This case is treated in the first part of Section 7 - 4, but the word parameter is not used in the text.)

If 2 first degree equations are given there are 3 possibilities:

1. In the most interesting case there are points (x,y,z) whose coordinates satisfy the 2 equations; the 2 equations impose 2 independent conditions on the point, and only one variable may be assigned arbitrarily. The point (x,y,z) now has only one degree of freedom. The number triples in the solution set of the system of 2 equations can be described in terms of a single parameter. The point is constrained to remain on a line--the line of intersection of the 2 given planes.

    Section 7 - 8 develops methods for describing the line of intersection in different ways, depending on which variable is assigned arbitrarily (which variable serves as parameter).

2. In the second case, the 2 equations are inconsistent. They represent parallel planes (discussed below and in starred section 7 - 7), and no number triple can satisfy both equations. Here there is no point that is common to both planes.
3. In the third case the 2 equations represent the same plane, and we have actually only 1 condition on the coordinates of points in this plane. Again the number triples in the solution set are described in terms of 2 parameters. (This case is also discussed below and in section 7 - 7.)

If a third first degree equation, consistent with the first two and independent of them, is given, an additional condition is imposed on the number triple (x,y,z). In this case, no variable may be chosen arbitrarily. The coordinates (x,y,z) are completely determined, and the point (x,y,z) is the single point of intersection of the three planes. This is one of the cases studied in Section 7 - 9. The cases in which the systems are dependent reduce to one of the two cases studied above: a line of intersection (one degree of freedom)--one parameter, or a plane of intersection (two degrees of freedom)--two parameters. If the system is inconsistent (and this can happen in a variety of ways, as illustrated in Figure 7 - 9b), then there is again no point that is common to the three planes.

## General Comments: An Outline of our Procedure--Suggestions to the Teacher.

We give here an outline of our procedure in this chapter and an indication, in certain parts, of teaching techniques that may make the presentation easier or of aspects of the problem that are not developed in the text but may be useful for the teacher to know.

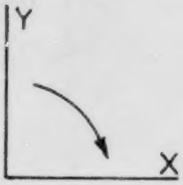
### The Purpose of the First Three Sections (Sections 7 - 1, 7 - 2, 7 - 3).

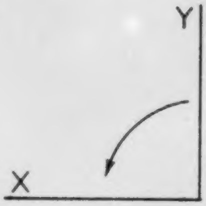
The first three sections are included in order to establish our basic geometric-algebraic correspondence; namely, the theorem that the equation of a plane is always of first degree, and that a first degree equation always represents a plane. There are several points that should be made here:

7 - 1. Comments. The coordinate system (Section 7 - 1) The coordinate system used is a "right-handed" one. This is an arbitrary choice that is made throughout this book because of the widespread use of this system in physics, vector analysis, and other courses in mathematics and its applications.

What do we mean by a "right-handed" or a "left-handed" system?

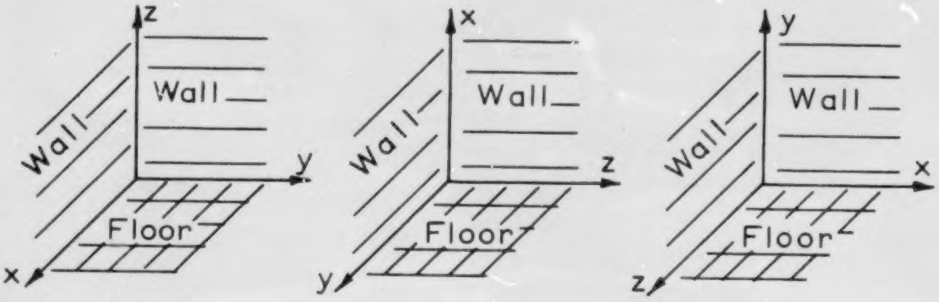
The difference between these systems may be expressed somewhat picturesquely perhaps, as follows: In a right-handed coordinate system, a person impaled on the positive Z-axis and looking toward the XY plane views it just as he always did in plane geometry--the positive X axis positive to the *right* of the positive Y axis. In a left-handed system our observer on the positive Z axis, on looking toward the XY plane sees the positive X axis extending to the *left* of the positive Y axis.



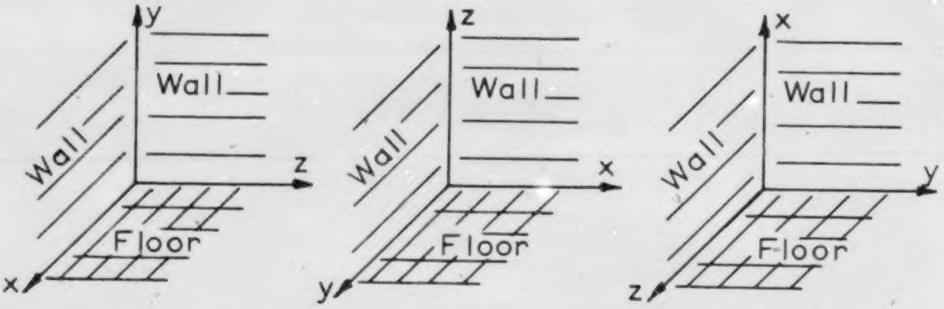


The following sketches illustrate a variety of the views an observer may have of each type of system.

Right Handed System


z
Wall
Wall
y
Floor
x
x
Wall
Wall
z
Floor
y
y
Wall
Wall
x
Floor
z

Left Handed System


y
Wall
Wall
z
Floor
x
z
Wall
Wall
x
Floor
y
x
Wall
Wall
y
Floor
z

7 - 2. Comments. The distance formula in space. In teaching the distance formula in space, many teachers have found that a box or other model constructed with pieces of hardware, cloth, window screen, or ordinary cardboard is very helpful. Even the corner of a room can be used to assist the student to visualize this, and other parts of geometry in space. The industrial arts teacher can be very helpful in providing large drawings for display purposes; and it may be possible to secure strip films that will show figures in three dimensions.

7 - 3. Comments. The correspondence between planes and first degree equations in three variables. In proving the theorem establishing the correspondence between planes in space and first degree equations in 3 variables (Section 7 - 3), we did not have available the customary techniques of solid analytic geometry for deriving the equation of a plane. Thus, instead of viewing the plane as the locus of points on lines perpendicular to a given line through a point on that line, we have adopted a different definition. We have viewed the plane as the locus of points equidistant from two given points. This definition enables us to derive the equation of the plane with no analytic machinery beyond the distance formula. Since our definition embodies a property that characterizes a plane, the equation we derive represents precisely the plane with all the properties studied in geometry. In particular, a pair of distinct planes are either parallel or they intersect in a line.

If the teacher is pressed for time, it is suggested that the proof in Section 7 - 3 be omitted. The student should then accept without proof the theorem that every plane in three dimensions can be represented by an equation of the form

$$ax + by + cz + d = 0$$

where a, b, c, d, are real constants, and a, b, c, are not all zero; and the converse theorem, that every equation of this form represents a plane.

7 - 4, 7 - 5. Comments. The graph of a first degree equation in three variables. The correspondence between a plane and a first degree equation is introduced to throw light on the algebraic problem. The ability to draw the graph of an equation will enable the student to gain insight into some of the special situations that may occur when, in the later sections, we study the solution sets of systems of two or three equations. In Sections 7 - 4 and 7 - 5 we try to develop this ability to draw graphs, first for the special planes that are parallel to an axis but not parallel to a coordinate plane (one variable has a zero coefficient, e.g., $x + y = 4$); second for planes parallel to a coordinate plane (two variables have zero coefficients--the equation gives a constant value for one coordinate, e.g., $x = 3$); and last for planes that have equations with no coefficients equal to zero. In all these cases, we consider the trace of the plane in each of the coordinate planes. (The trace is the intersection

of the given plane with a coordinate plane.) The trace, like every line, is described by two first degree equations, but one of these is the equation of a coordinate plane, i.e., $x = 0$ or $y = 0$ or $z = 0$.

## Some Generalizations About Planes and Their Equations--Suggestions for Constructing Problems.

We are now in a position to make certain general observations:

1. Any plane has infinitely many equations.

If a given plane is represented by the equation

$$ax + by + cz + d = 0,$$

it is also represented by

$$k(ax + by + cz + d) = 0,$$

where k is any non-zero constant.

The proof of this may be either algebraic or geometric:

(a) every number triple satisfying either equation satisfies the other; or

(b) the traces of the two planes are identical.

The converse is also true. Equations in which the coefficients are proportional represent coincident planes. For if two planes have equations

$$ax + by + cz + d = 0$$

$$Ax + By + Cz + D = 0$$

and $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \frac{d}{D} = k$

then the first equation is k times the second, and the equations represent the same plane.

These results are useful throughout our algebraic study. For example, if we have a system of two equations, in which one equation is a multiple of the other, we know that the second equation contributes no information not already given by the first. Thus a point whose coordinates satisfy the two equations still has the two degrees of freedom that characterize the point whose coordinates satisfy a single equation. This is the third case described under Degrees of Freedom for two equations.

2. If two planes have equations that can be reduced to the form

$$ax + by + cz = d$$

$$\text{where } d \neq f$$

$$ax + by + cz = f$$

the planes are parallel, and there is no common solution.

Again, the proof may be algebraic or geometric:

(a) If $(x_0, y_0, z_0)$ is any number triple, it cannot satisfy both these equations since

$$ax_0 + by_0 + cz_0$$

cannot be equal to both d and f if $d \neq f$; or

(b) the traces of the planes are parallel lines.

3. These two cases are summarized in the following rule:

   *If corresponding coefficients of two first degree equations are proportional, then their graphs*

   (a) *coincide if the constant terms have the same ratio as the coefficients,*

   (b) *are parallel if their constant terms are not in the same ratio as the coefficients.*

This information gives us a way to recognize at a glance two equations that are inconsistent or dependent. It also gives the teacher the ability to make up problems with great ease. He need merely put down any left member of first degree and any constant term for the first equation. For the first case, double the first equation, triple it, or transpose some terms. For example:

Start with the equation $3x + 5y - z = 7$.

Double the equation and transpose the y term

$$6x - 2z = 14 - 10y .$$

The resulting equations represent coincident planes. For the second case, copy down the same left member for the equation but make its constant term different:

$$3x + 5y - z = 12 .$$

This equation represents a plane parallel to the first two. A more sophisticated version of this procedure involves doubling, trebling, or changing the sign of the left member while taking care to do something else with the constant.

4. Conversely, two planes meet in a line if and only if their corresponding coefficients are not proportional.

   Again, examples of this sort can be invented in the time it takes to write them down: take any first degree equation as the first, and change the coefficients for the second one somehow, so that they are not proportional to the first ones. Having accomplished this much one is safe. Any constant term whatever will do. New coefficients not proportional to the first ones can be obtained in many ways. For example: keep one of them the same and change some other one; or add one to each of them; or change some of the signs, but not all, etc. We give a collection of such equations:

$$3x + 5y - z = 7$$

$$3x + 5y + z = 7$$

$$4x - 5y + 2z = 5, \text{ etc.}$$

The four results stated above may be made the basis of a preliminary examination of a system of equations. If we can tell by inspection that two of the given equations are inconsistent, we know immediately that the system has no solution. If we can tell by inspection that one of the given equations is dependent on the others, we know that the number of degrees of freedom is larger than would be the case if all the equations were independent.

7 - 7. *Comment*. The geometric information presented in Sections 7 - 4 and 7 - 5 is sufficient for the minimal purposes of the chapter. Section 7 - 7 presents additional geometric information that will throw light on the later work and will be of interest to those students who enjoy three-dimensional studies. If the teacher finds time to include this material, and if some students find the graphing of space figures excessively difficult, it may be helpful if such assignments are made to groups of 2 or 3 students. If section 7 - 7 is omitted, *the teacher should be sure to teach the material presented in Examples 2 and 3 of the section. This is covered by the discussion summarized in point (3) above.*

### *Parametric Representation of the Line of Intersection of Two Intersecting Planes.*

(a) *The line intersects all the coordinate planes.*

Once we have disposed of systems of 2 equations in 3 variables in which the planes are coincident or parallel, we must undertake the more formidable problem of representing the line of intersection of planes that do intersect. Actually *this line is represented by the two equations of the given intersecting planes*; but since we know from our discussion above that the point describing the line has a single degree of freedom, we seek a representation of the line in which the three coordinates of the point are described in terms of a single parameter. We seek to describe the coordinates of any point on the line as functions of a single variable--this is the variable to which we can assign arbitrary values in finding as many points as we want in the solution set. Our manipulation of the given equations is aimed at expressing all three variables in terms of one of them so that the variable that is arbitrary is clearly indicated. In the non-special case, when the line of intersection cuts all three coordinate planes, any one of the three variables may be chosen arbitrarily, so that there are three different parametric representations of the line, one in which x is arbitrary, one in which y is arbitrary, and one in which z is arbitrary. To derive each of these, we find the equations of a pair of planes through the line, each equation having in it one variable with a zero coefficient. This is achieved by eliminating each variable in turn from the given equations, and combining the resulting three equations, two by two. For example, an equation containing only x and y (the coefficient of z is zero) is combined with an equation containing only x and z. Since x and y are in the first equation, y can be expressed in terms of x.

Since x and z are in the second equation, z can be expressed in terms of x. In this case, x serves as parameter. (Note that geometrically the planes corresponding to equations that have a single zero coefficient are parallel to an axis, e.g., Equation 7 - 8d is $x + 3z + 1 = 0$; this plane is parallel to the y-axis.)

(b) The line is perpendicular to one of the coordinate planes.

If the line we seek to describe is perpendicular to one of the coordinate planes the situation is special. For example, if the line is perpendicular to the XY-plane, the plane that passes through it and is parallel to the x-axis is also parallel to the XZ-plane. Similarly, the plane that passes through the given line, parallel to the y-axis, is also parallel to the YZ plane. Indeed any plane that passes through the given line is parallel to the z-axis. Thus the coordinates of a point on the line have a very special parametric representation, namely,

$x = a$(y and z have zero coefficients)
$y = b$(x and z have zero coefficients)
z is arbitrary

(c) The line is parallel to one of the coordinate planes. In a similar way, we find that if the line we seek to describe is parallel to one coordinate plane, but intersects the other two, the situation is special. Here, one variable will be constant, but either of the others may be expressed in terms of the third. This case is discussed in Example 2 of Section 7 - 8.

## The Method of Elimination

The justification for the familiar procedure used in eliminating one variable from the given equations is the theorem that is illustrated for a special case in starred Section 7 - 10. It establishes the fact that the solution set for any given system

$$\begin{cases} f_1 = 0, \\ f_2 = 0; \end{cases}$$

is the same as for a new system

$$\begin{cases} f_1 = 0, \\ a_1 f_1 + a_2 f_2 = 0. \quad (a_1 \text{ and } a_2 \text{ not both zero}) \end{cases}$$

or for a second system

$$\begin{cases} a_1 f_1 + a_2 f_2 = 0, & \text{Eq. (1)} \\ b_1 f_1 + b_2 f_2 = 0. \quad (b_1 \text{ and } b_2 \text{ not both zero}) & \text{Eq. (2)} \end{cases}$$

(The two new systems are said to be equivalent to the given system.) The expressions in the left members of Equations (1) and (2) are *linear combinations* of $f_1$ and $f_2$. Equations (1) and (2) represent planes through the line of intersection of the planes of the given system if these intersect; they represent planes parallel to the given planes if these are parallel; they represent planes coinciding with the given planes if these coincide. It is the case for intersecting planes that is studied for a special pair of equations in Section 7 - 10. (The teacher is urged to study this section, even if it is not covered in class, to gain some familiarity with these ideas.)

7 - 8. *Comments*. *Application of the method of elimination*.

In Section 7 - 8 the equations we obtain represent planes through the line of intersection we seek to describe. Therefore when we use the new equations to write in parametric form the coordinates of a point on the line of intersection, we have actually described a point on the line of intersection of the *given* planes as well as on the line of intersection of the planes represented by the new equations.

7 - 9. *Comments*. *Application of the method of elimination*.

The same idea dominates Section 7 - 9. Consider an example studied there. We discussed the system (Example 1)

$$\begin{aligned} x + 2y - 3z &= 9 \\ 2x - y + 2z &= -8 \\ -x + 3y - 4z &= 15 \end{aligned}$$

and converted it into the equivalent system

$$\begin{aligned} x + 2y - 3z &= 9 \\ -5y + 8z &= -26 \\ z &= -2 \end{aligned}$$

by repeated application of exactly the same technique used in Section 7 - 8: selecting two of our equations and playing them off against one another to get rid of variables one at a time. Even the final stage of the discussion of Example 1 is an instance of the same process. We arrive eventually at the system

$$\begin{aligned} x &= -1 \\ y &= 2 \\ z &= -2 \end{aligned}$$

by subtracting appropriate multiples of the last equation from the first two, eliminating z, and then using the second equation to get y out of the first. This leaves us with the last system given above which is equivalent to the original system. The last system is so extraordinarily simple that we can read off its solution set at a glance.

## A Systematic Method for Studying Three Equations in Three Variables

The problem discussed in Section 7 - 9 is the most complicated case we consider with three variables--the case in which there are as many equations as variables. Figure 7 - 9b illustrates the eight essentially different configurations formed by three planes in space. These pictures are included only for their interest. It is not important at this point that the student understand all the details.

With three planes there are four different types of solution sets (there were only three in the case of two planes):

1. The empty set
2. A single point
3. A line
4. A plane

The main business of Section 7 - 9 is the presentation of a systematic algebraic method for determining everything there is to know about systems of first degree equations: whether there are any solutions and how to find all of them. This method, "elimination", is applicable to systems having any number of equations and any number of variables. It is spelled out in detail only for three equations in three variables, since this case is probably the smallest one complicated enough to be of any real interest. Restricting ourselves to this case, we give examples to illustrate the application of the method not only to the type of system in which the solution set consists of a single number triple, but also to several types in which the systems are inconsistent or dependent.

This method (sometimes called triangulation) is attributed to Gauss (1777-1855), the greatest mathematician since Newton. It gives the student the basic point of view he will need if he goes on to work in a large computing center. Its popularity reflects the fact that it gives an orderly procedure for handling systems of linear equations which, for many important cases, involves substantially fewer arithmetic operations than other methods. Using this method, we have the system essentially solved by the time we discover whether or not the solution is unique.

Relation of Method of Elimination to Cramer's Rule.

Those familiar with Cramer's rule (this is discussed in many of the older texts on College Algebra; it is usually not included in the newer texts) may be interested in its relation to the method of Gauss that we have presented. Observe first that, if the given equations are

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

then our "triangulation" method replaces the given system by

$$\left\{ \begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} z = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} y + \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} z = \begin{vmatrix} a_1 & d_1 \\ a_3 & d_3 \end{vmatrix} \end{array} \right.$$

and then by

$$\begin{array}{llll} a_1x & + b_1y & + c_1z & = d \\ & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y & + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} z & = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} \end{array}$$

$$\begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \end{vmatrix} z = \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_3 & d_3 \end{vmatrix} \end{vmatrix}$$

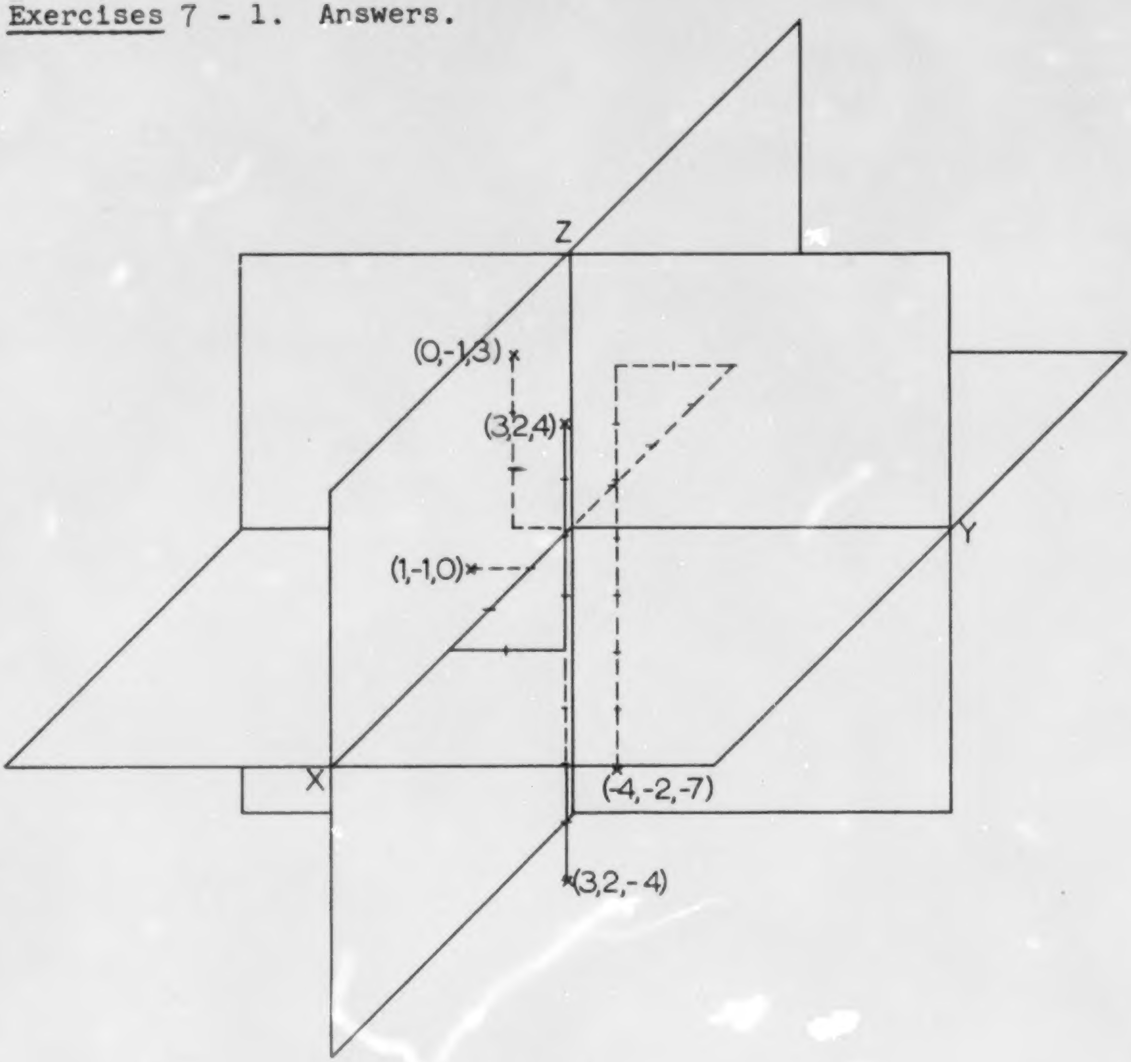
The last equation can be shown to be equivalent to

$$a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} z = a_1 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

This is one of the equations derived by applying Cramer's rule. But for practical computing, the "elimination" or "triangulation" method has the great advantage that the nature of the solution becomes clear at this point; if it is unique, we find the solution with a minimum of additional computation. The mastery of this method should be a principal objective in teaching the chapter.

Summary. We summarize by pointing out that, throughout Chapter 7 the manipulations that enable us to find the solution set for a system of equations are justified by the fact that the given system is consistently replaced by an equivalent system. The new system is equivalent to the old because the new equations are derived by taking linear combinations of the expressions defining the given equations. Hence the planes defined by the new equations pass through the intersection of the given planes; when we have described the solution set of the new equations, we have also described the solution set of the given equations.

Exercises 7 - 1. Answers.



The points in Exercises (1), (3), (5), (7), (9) are represented on the graph.

11. (a) On the yz plane
    (b) On a plane || to the yz plane cutting the x axis at 2.
    (c) On a plane || to the yz plane cutting the x axis at -3.
12. (a) On the xz plane
    (b) On a plane || to the xz plane and cutting the y axis at 3.
13. (a) On a plane || to the xy plane and cutting the z axis at 2.
    (b) On a plane || to the xy plane and cutting the x axis at -2.
14. A plane || to the z axis and cutting the x and y axis at 4.

Exercises 7 - 2. Answers.

1. $4\sqrt{2}$
2. 7
3. 5
4. 12
5. 3
6. $\sqrt{129}$
7. $\sqrt{41}$
8. $\sqrt{14}$

Exercises 7 - 3. Answers.

1. $10x - 10y - 8z - 10 = 0$

   $2x - 6y - 12z + 6 = 0$

   $-10x + 4y - 8z = 0$

2. $4x + 4y - 16z = 32$

   $6x + 8y - 6z = -14$

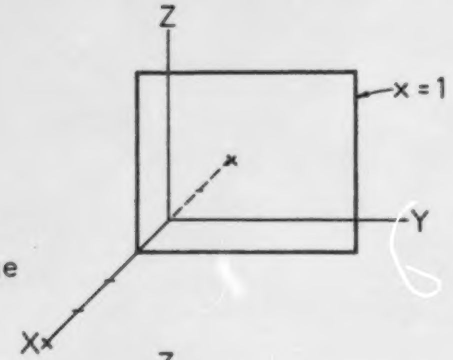
   $4x - 8y + 12z = 0$

3. (a) $(4,0,0)$ , $(-2,0,0)$.

   Plane has equation

   $x = 1$

   Plane is parallel to YZ plane and cuts the X axis at 1.

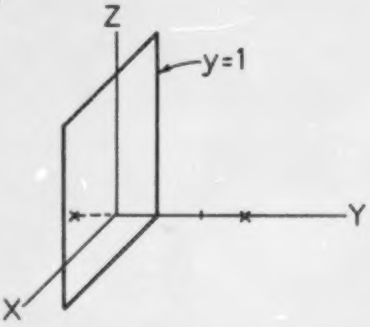


   (b) $(0,3,0)$ , $(0, -1,0)$.

   Plane has equation

   $y = 1$

   Plane is parallel to XZ plane and cuts the Y axis at 1.



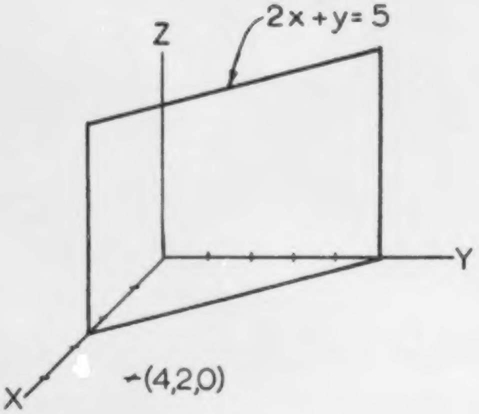
   (c) $(0,0,0)$ , $(4,2,0)$.

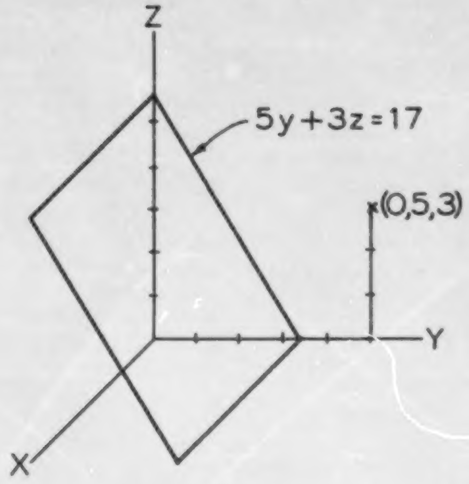
   Plane has equation

   $8x + 4y = 20$

   or $2x + y = 5$

   Plane is parallel to the Z axis, and cuts the X axis at $\frac{5}{2}$ and the Y axis at 5.





(d) (0,0,0) , (0,5,3).

Plane has equation

$$10y + 6z = 34$$

or $5y + 3z = 17$

Plane is parallel to the X axis, and cuts the Y axis at $\frac{17}{5}$ and the Z axis at $\frac{17}{3}$

4. (a) Let $P(x,y,z)$ be any point on the plane that is the locus of points equidistant from $A(0,0,0)$ and $B(ka,kb,kc)$ where

$$k = \frac{-2d}{a^2 + b^2 + c^2}.$$

Then $PA = PB$

$$x^2 + y^2 + z^2 = (x - ka)^2 + (y - kb)^2 + (z - kc)^2$$

$$0 = -2kax + k^2a^2 - 2kby + k^2b^2 - 2kcz + k^2c^2$$

$$2k(ax + by + cz) = k^2(a^2 + b^2 + c^2)$$

$$ax + by + cz = \frac{k}{2}(a^2 + b^2 + c^2)$$

Put $k = \frac{-2d}{a^2 + b^2 + c^2}$

The equation of the locus becomes

$$ax + by + cz = -d$$

Thus $ax + by + cz + d$ represents the required locus. If $d = 0$, it follows that $k = 0$. The two points coincide, and no plane is determined. The case where $d = 0$, is treated in part (b).

(b) Let P(xyz) be any point on the plane that is the locus of points equidistant from A(a,b,c) and B(-a,-b,-c). Since these points are symmetric with respect to the origin, we would expect the required plane to pass through the origin.

$$PA = PB$$

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = (x + a)^2 + (y + b)^2 + (z + c)^2$$

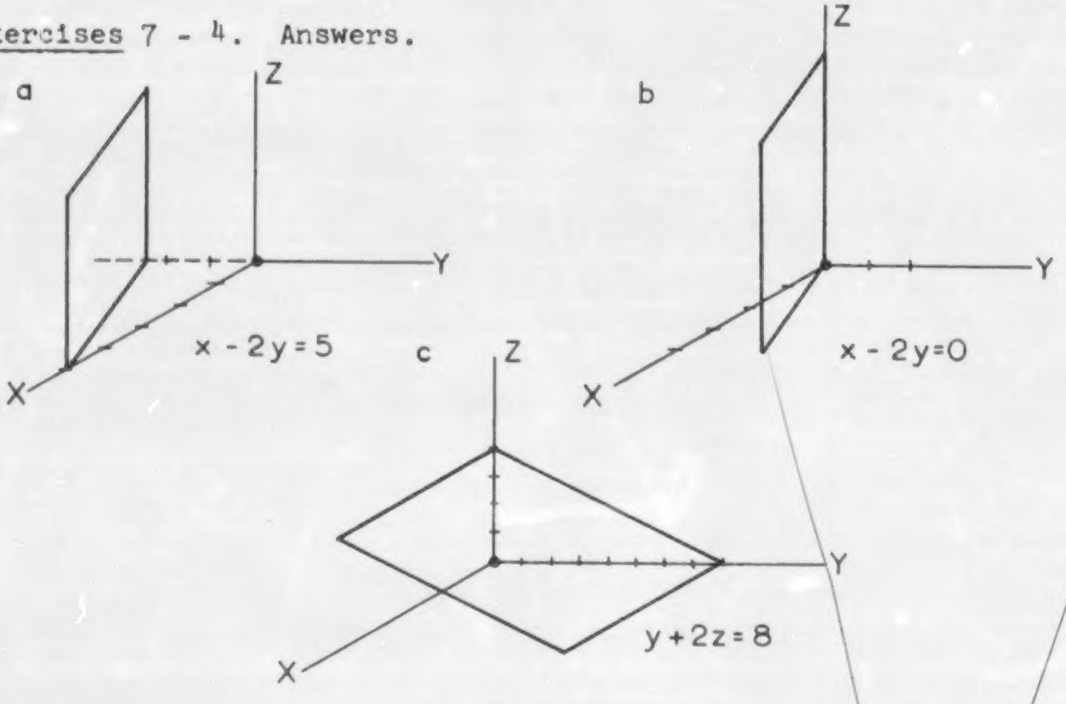
$$-2ax - 2by - 2cz + a^2 + b^2 + c^2 = 2ax + 2by + 2cz + a^2 + b^2 + c^2$$

$$4ax + 4by + 4cz = 0$$

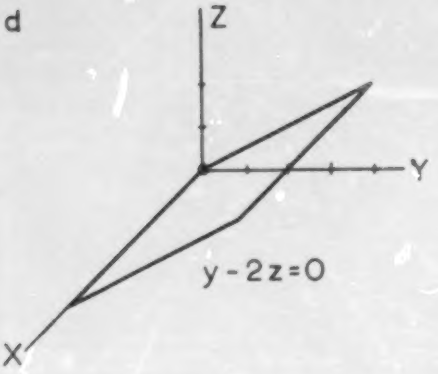
$$ax + by + cz = 0$$

Since (0,0,0) is a point in the solution set of this equation, the plane passes through the origin.

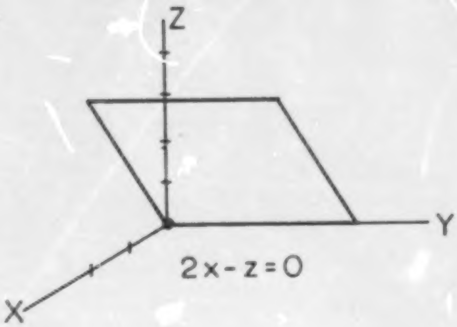
Exercises 7 - 4. Answers.



d



e



2. $2x + y = 6$

A(3,0,0). Also on the graph are

(3,0,1) , (3,0,2) , (3,0,4).

B(1,4,0). Also on the graph are

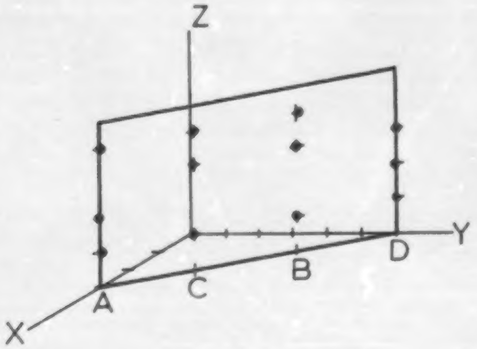
(1,4,1) , (1,4,3), (1,4,4).

C(2,2,0). Also on the graph are

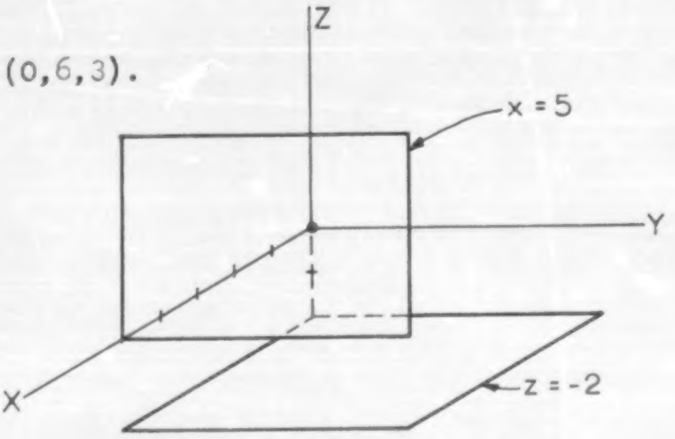
(2,2,1) , (2,2,3) , (2,2,4).

D(0,6,0). Also on the graph are

(0,6,1) , (0,6,2) , (0,6,3).

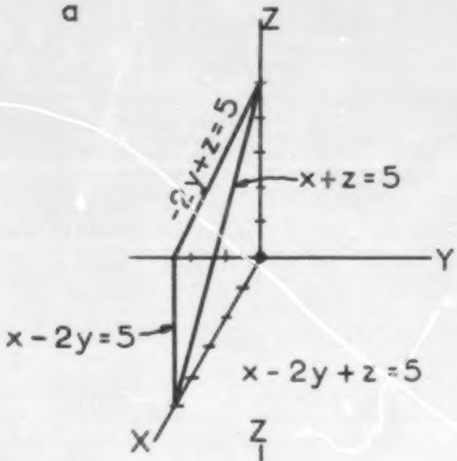


3.

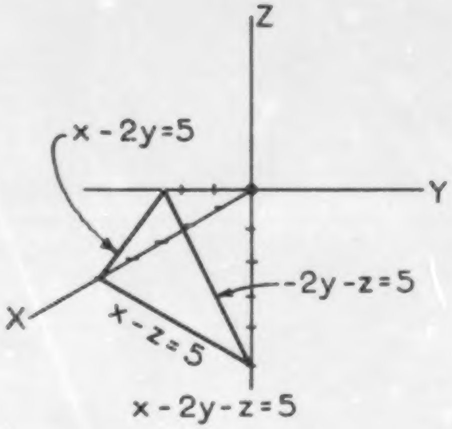


Exercises 7 - 5. Answers.

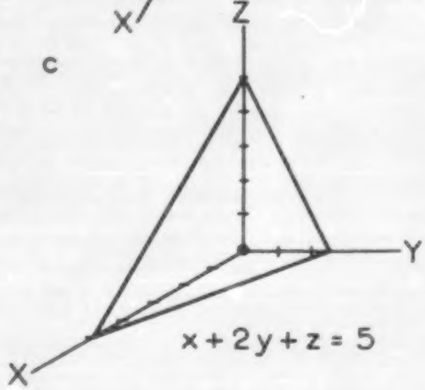
1. a


Z
-2y+z=5
x+z=5
Y
x-2y=5
x-2y+z=5
X

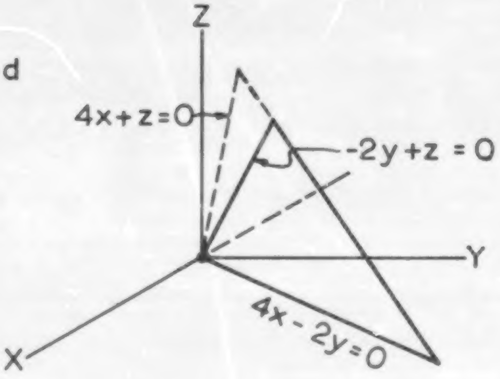
b


Z
x-2y=5
Y
-2y-z=5
X
x-z=5
x-2y-z=5

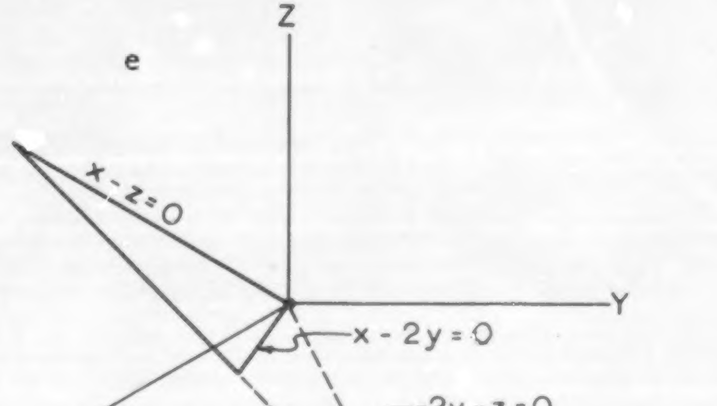
c



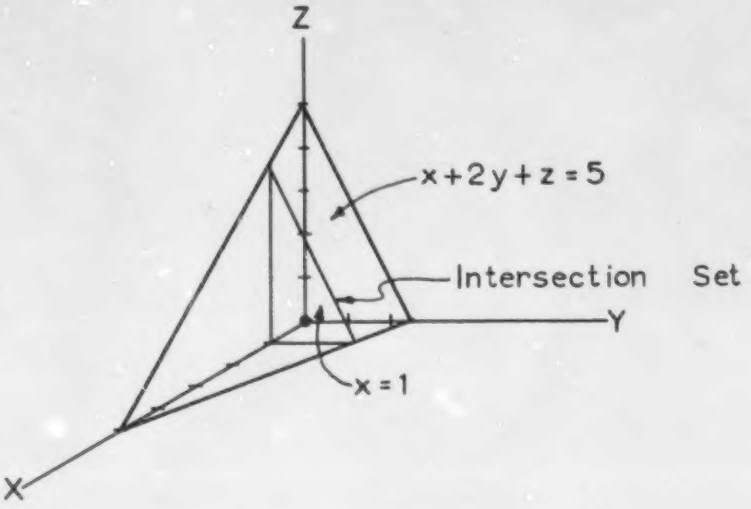
d


Z
4x+z=0
-2y+z=0
Y
4x-2y=0
X

e

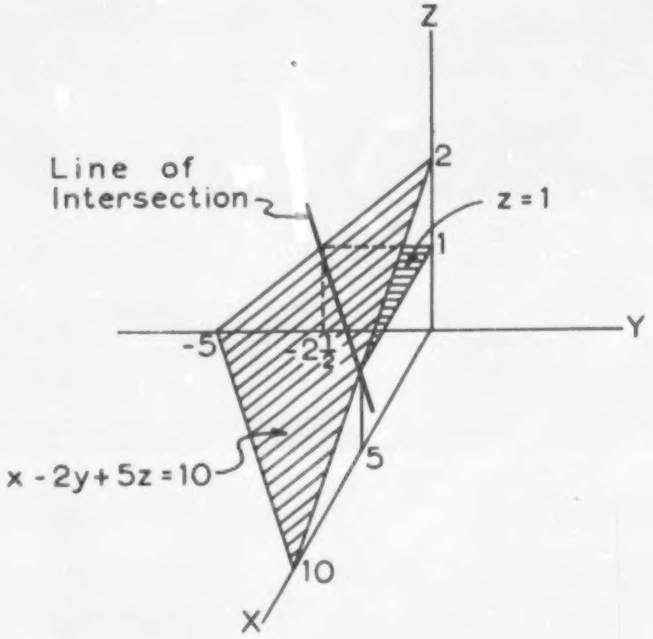

Z
x-z=0
Y
x-2y=0
-2y-z=0
X
x-2y-z=0

*2.


Z
x+2y+z=5
Intersection Set
Y
x=1
X

Exercises 7 - 7. Answers.

1.


Z
2
Line of Intersection
z=1
1
Y
-5
$-2\frac{1}{2}$
5
x-2y+5z=10
10
X

$x - 2y + 5z = 10$

Intercepts: (10,0,0)

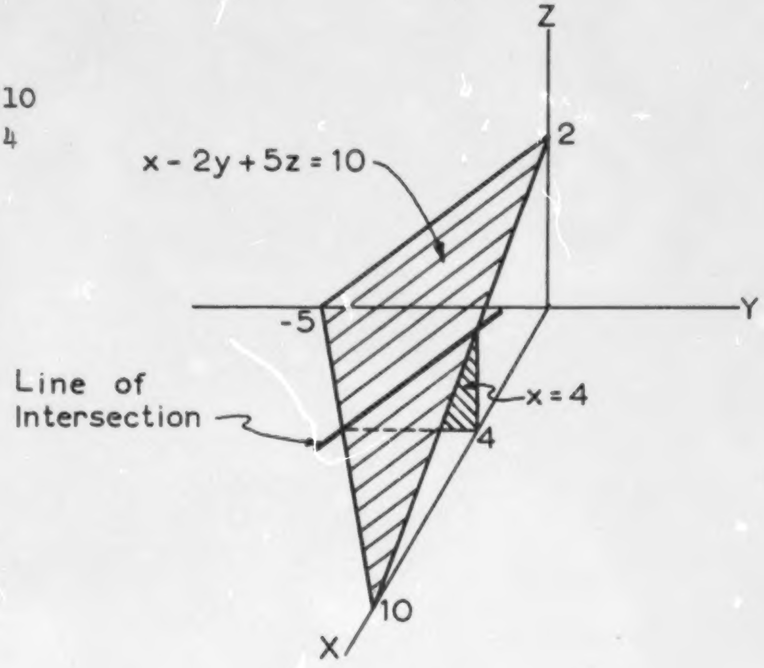
(0,-5,0)

(0,0,2)

Z = 1 Parallel to XY plane.

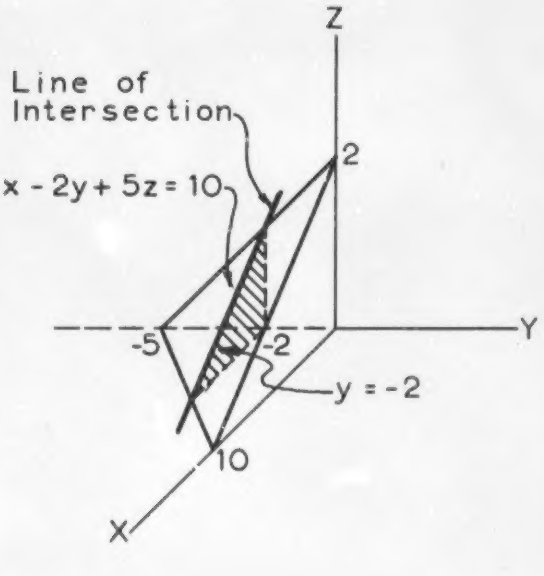
2. $x - 2y + 5z = 10$

$x = 4$


Z
Y
X
2
-5
4
10
x - 2y + 5z = 10
x = 4
Line of Intersection

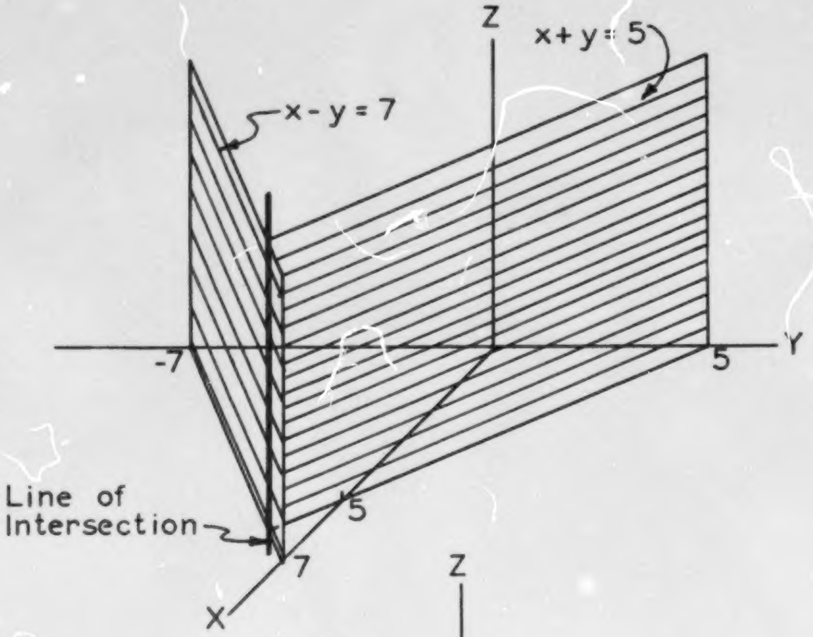
3. $x - 2y + 5z = 10$

$y = -2$


Z
Y
X
2
-5
-2
10
Line of Intersection
x - 2y + 5z = 10
y = -2

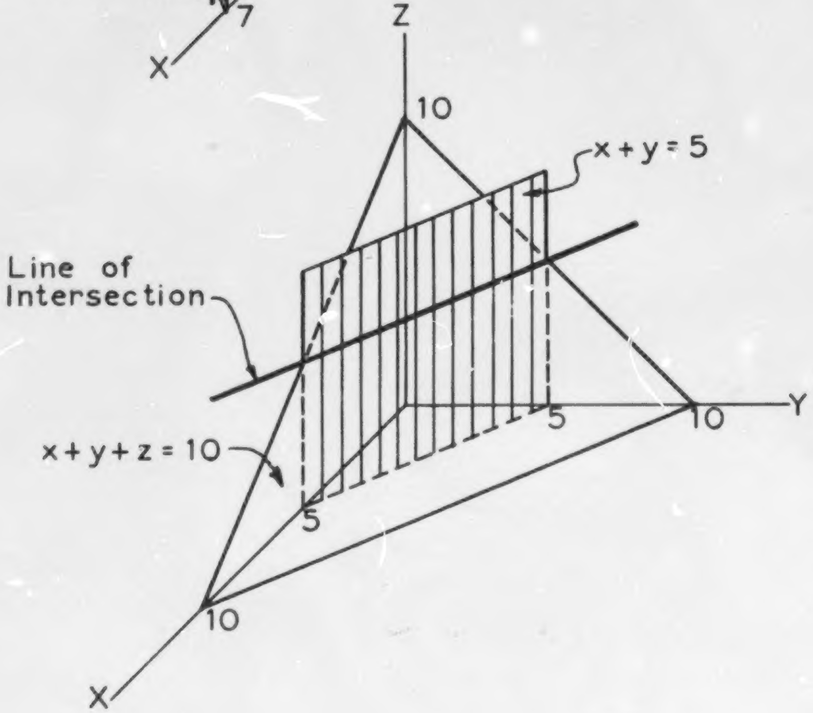
4. x + y = 5

x - y = 7


Z
x+y=5
x-y=7
-7
Y
5
Line of Intersection
5
7
X

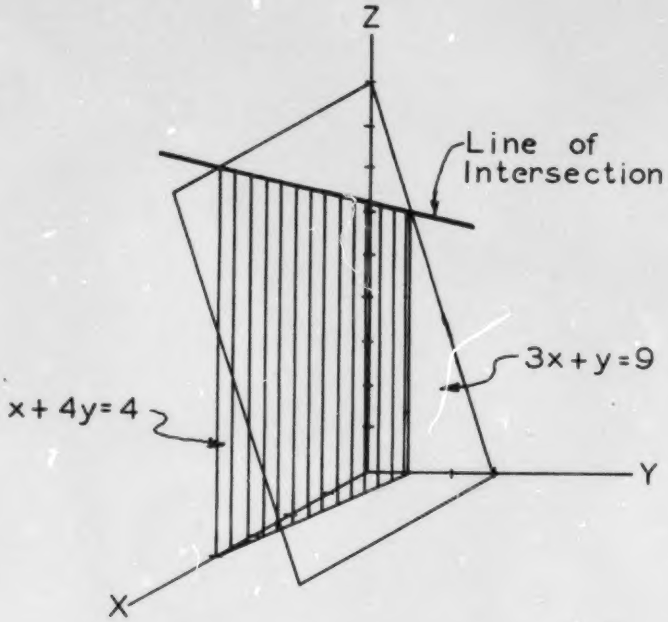
5. x + y = 5

x + y + z = 10


Z
10
x+y=5
Line of Intersection
5
10
Y
x+y+z=10
5
10
X

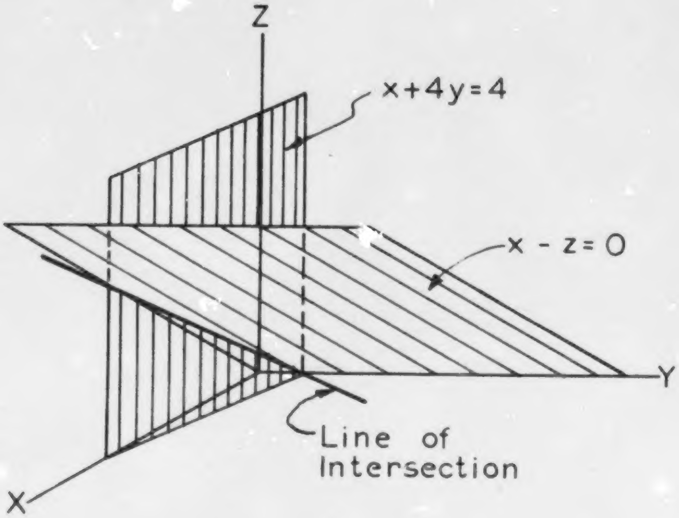
6. $3y + z = 9$

$x + 4y = 4$


Z
Line of Intersection
3x+y=9
x+4y=4
Y
X

7. $x + 4y = 4$

$x - z = 0$


Z
x+4y=4
x - z=0
Y
Line of Intersection
X

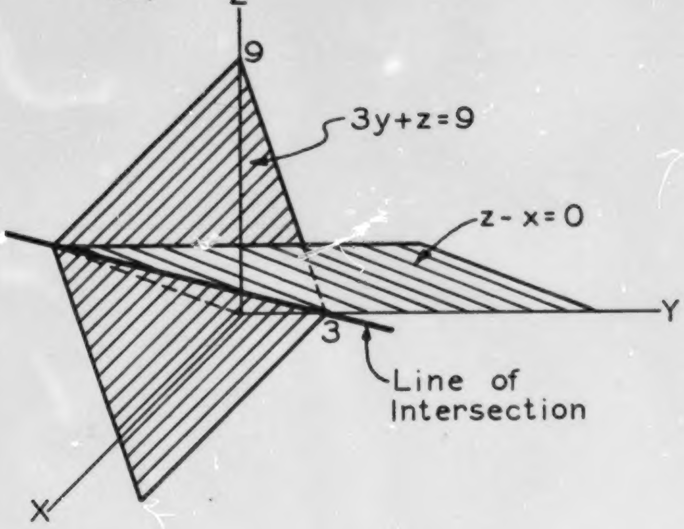
8. $3x + y - z = 2$

$2z = 6x + 2y - 4$

Same plane

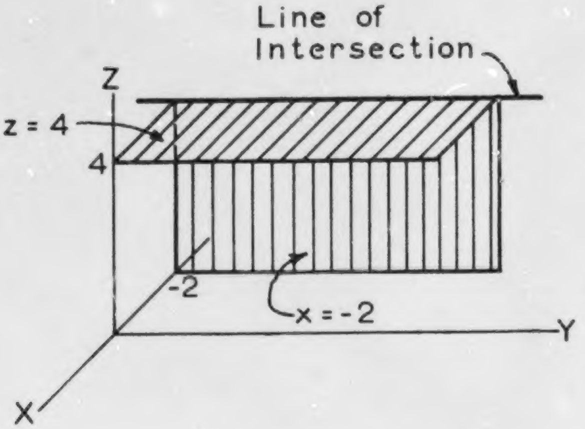
9. $z - x = 0$

$3y + z = 9$


Z
9
3y+z=9
z-x=0
Y
3
Line of Intersection
X

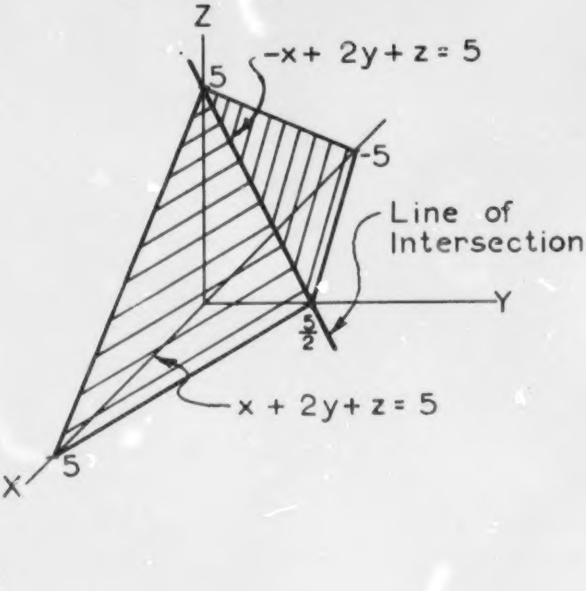
10. $x = -2$

$z = 4$


Line of Intersection
Z
z = 4
4
-2
x = -2
Y
X

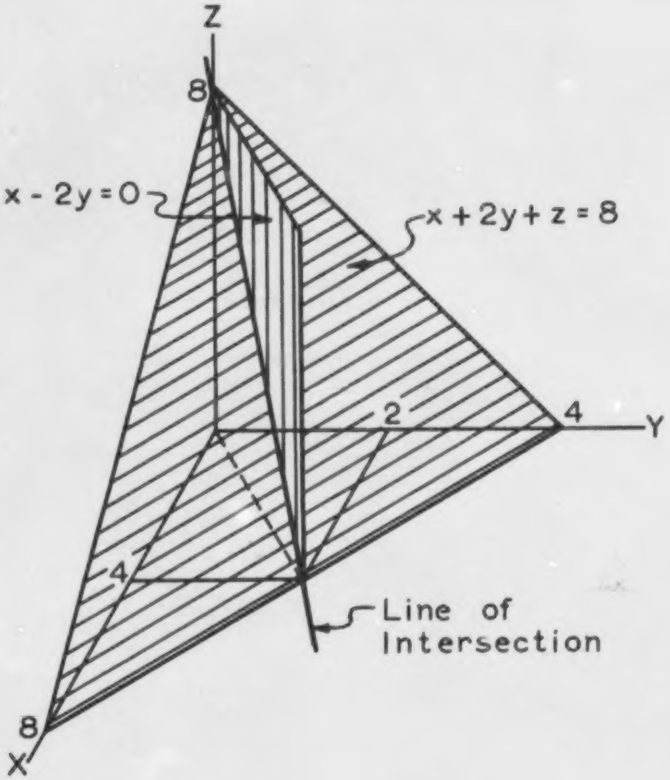
11. $x + 2y + z = 5$

$-x + 2y + z = 5$


Z
5
-x + 2y + z = 5
-5
Line of Intersection
Y
5/2
x + 2y + z = 5
5
X

12. $x + 2y + z = 8$

$x - 2y = 0$


Z
8
x - 2y = 0
x + 2y + z = 8
2
4
Y
4
Line of Intersection
8
X

Exercises 7 - 8. Answers.

1. $x + 2y - z = 8,$

$x + y + z = 0.$

$y - 2z = 8$

$2x + 3y = 8$

$\therefore x = \frac{1}{2}(-3y + 8)$

y arbitrary

$z = \frac{1}{2}(y - 8)$

| x | 4 | 1 | -2 | -5 |
|---|---|---|---|---|
| y | 0 | 2 | 4 | 6 |
| z | -4 | -3 | -2 | -1 |

Check by substituting in the given equations.

$-3y + 8 + 4y - y + 8 \overset{?}{=} 16$

$-3y + 8 + 2y + y - 8 \overset{?}{=} 0$

2. $x + y - z = 5$

$x + 2y + z = 0$

$y + 2z = -5$

$2x + 3y = 5$

$x = \frac{1}{2}(-3y + 5)$

$y$ arbitrary

$z = \frac{1}{2}(-y - 5)$

| | | | | |
|---|---|---|---|---|
| x | $\frac{5}{2}$ | 1 | -2 | -5 |
| y | 0 | 1 | 3 | 5 |
| z | $-\frac{5}{2}$ | -3 | -4 | -5 |

Check by substituting in the given equations.

$-3y + 5 + 2y + y + 5 \overset{?}{=} 10$

$-3y + 5 + 4y - y - 5 \overset{?}{=} 0$

3. $-2x + y + 3z = 0,$

$-4x + 2y + 6z = 0.$

Since the second equation is twice the first, the two planes coincide. Thus there is no line of intersection.

4. $3x - 4y + 2z = 6,$

$6x - 8y + 4z = 14.$

If we divide both members of the second equation by 2, we obtain

$$3x - 4y + 2z = 7$$

We can see by inspection that no number triple can be in the solution set of both these equations. Therefore the corresponding planes have no point in common. The planes are parallel.

5. $4x - 7y + 6z = 13,$

$5x + 6y - z = 7.$

$-59y + 34z = 37$

$34x + 29y = 55$

$x = \frac{1}{34}(-29y + 55)$

$y$ arbitrary

$z = \frac{1}{34}(59y + 37)$

| | | | | |
|---|---|---|---|---|
| x | $\frac{55}{34}$ | $\frac{13}{17}$ | $\frac{42}{17}$ | $-\frac{3}{34}$ |
| y | 0 | 1 | -1 | 2 |
| z | $\frac{37}{34}$ | $\frac{48}{17}$ | $\frac{-11}{17}$ | $\frac{155}{34}$ |

Exercises 7 - 9. Answers.

1. (2,-1,1).
2. (1,-1,2).
3. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = -\frac{1}{7}(6z - 5)$$

$$y = +\frac{1}{7}(16z - 11)$$

$z$ arbitrary

4. The three planes coincide. The solution set is an infinite set of triples corresponding to all the points in the plane.
5. The system is inconsistent. The solution set is empty.
6. (1,-1,1)
7. $(\frac{3}{4}, \frac{1}{2}, 3)$
8. (-1,-2,3)
9. $(\frac{1}{3}, -\frac{2}{5}, \frac{1}{2})$

10. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = 2z$$

$$y = \frac{1}{2}z$$

$$z \quad \text{arbitrary}$$

11. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{1}{5}(-7z + 17)$$

$$y = \frac{1}{5}(z - 1)$$

$$z \quad \text{arbitrary}$$

12. The system is inconsistent. The solution set is empty.

13. $(\frac{1}{3}, \frac{1}{2}, 1)$

14. The system is inconsistent. The solution set is empty.

15. The three planes have a line in common. (The second equation represents the same plane as the first equation.) The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{9}{2} - y$$

$$y \quad \text{arbitrary}$$

$$z = -\frac{3}{2}$$

16. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$\begin{aligned} x &= \tfrac{1}{7}(-z + 2) \\ y &= \tfrac{1}{7}(-5z + 17) \\ z & \quad \text{arbitrary} \end{aligned}$$

17. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$\begin{aligned} x &= -7z - 10 \\ y &= -5z - 6 \\ z & \quad \text{arbitrary} \end{aligned}$$

18.

| | Vitamin Content | | |
|---|---|---|---|
| Food | A | B | C |
| I | 1 | 3 | 4 |
| II | 2 | 3 | 5 |
| III | 3 | 0 | 3 |
| Requirements | 11 | 9 | 20 |

If we buy x units of I, y of II, 20 of III, we want

$$\begin{array}{rcr} x + 2y + 3z = 11, & & x + 2y + 3z = 11, \\ 3x + 3y = 9, & \text{or} & x + y = 3, \\ 4x + 5y + 3z = 20; & & 4x + 5y + 3z = 20. \end{array}$$

Eliminate x:

$$\begin{aligned} x + 2y + 3z &= 11, \\ -y - 3z &= -8, \\ -3y - 9z &= -24; \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + 3z &= 11, \\ y + 3z &= 8, \\ y + 3z &= 8. \end{aligned}$$

Answer for (a): No--Our conditions are dependent.

for (b): Consider the system,

$$\begin{aligned} x + 2y + 3z &= 11, \\ y + 3z &= 8, \\ 6x + y + z &= 10. \end{aligned}$$

Eliminate x:

$$\begin{aligned} x + 2y + 3z &= 11, \\ y + 3z &= 8, \\ 11y + 17z &= 56. \end{aligned}$$

Eliminate y:

$$\begin{aligned} x + 2y + 3z &= 11, \\ y + 3z &= 8, \\ 16z &= 32. \end{aligned}$$

Thus $z = 2$, $y = 8 - 3z = 2$, $x = 11 - 2y - 3z = 1$

Answer for (b): Yes. 1 unit of I and 2 each of II, III.

19. $x + y - 5 = 0,$

$-x + 3z - 2 = 0,$

$x + 2y + z - 1 = 0,$

$y + z + 4 = 0.$

Suppose we apply the standard procedure given in Section 7-9, if only to see what happens. We eliminate x by subtracting appropriate multiples of the first equation from others:

$$\begin{aligned} x + y - 5 &= 0, \\ y + 3z - 7 &= 0, \\ y + z + 4 &= 0, \\ y + z + 4 &= 0. \end{aligned}$$

We have found that, in our original system, the first, third and fourth equations are dependent; indeed

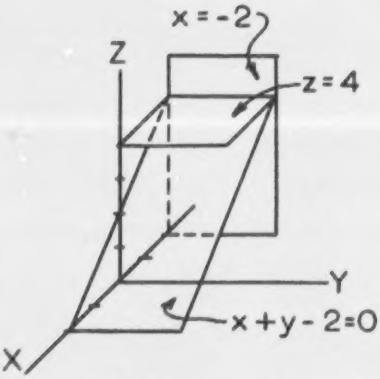
$$x + y - 5 = 1(x + 2y + z - 1) - 1(y + z + 4).$$

These three therefore all meet in a line. Since we were given the fact that the system has only one solution triple, this line must pierce the second plane in a single point. Hence any one of the four equations except for the second may be omitted, the line being determined by any pair of the three planes containing it.

Exercises 7 - 10. Answers.

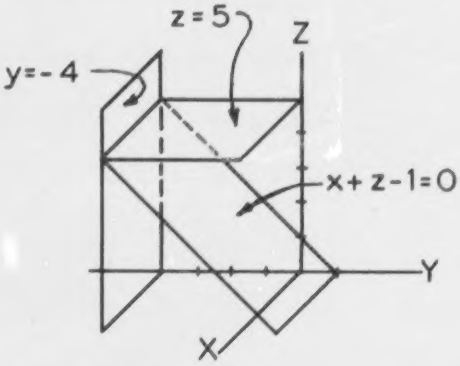
1. (a) $a(x + 2) + b(z - 4) = 0$

$a = 1,\ b = 1;\quad x + z - 2 = 0$



(b) $a(y + 4) + b(z - 5) = 0$

$a = 1,\ b = 1;\ y + z - 1 = 0$



2. (a) $a(x + 2y - 3z) + b(x - y + z - 1) = 0$

Substituting $(1,2,1)$ for $(x,y,z)$:

$a(1 + 4 - 3) + b(1 - 2 + 1 - 1) = 0$

$2a - b = 0 \quad ; \quad b = 2a$

Take $a = 1$, $b = 2$;

$(x + 2y - 3z) + 2(x - y + z - 1) = 0$

or $3x - z = 2$

(b) $a(2y - 3z - 2) + b(x + y + z) = 0$

Substituting $(3,-1,0)$ for $(x,y,z)$:

$a(-2 + 0 - 2) + b(3 - 1 + 0) = 0$

$-4a + 2b = 0$

$2a = b$

Take $a = 1$, $b = 2$

$(2y - 3z - 2) + 2(x + y + z) = 0$

$2x + 4y - z - 2 = 0$

(c) $a(x + z) + b(2x - y + z - 8) = 0$

Substituting $(0,0,0)$ for $(x,y,z)$:

$-8b = 0 \quad ; \quad b = 0.$

The equation is $x + z = 0$.

This shows that the plane represented by the first equation is the only plane through the given line of intersection that also passes through the origin.

(d) $a(2x - y + z - 3) + b(x - 3y + 4) = 0$

Substituting $(2,2,1)$ for $(x,y,z)$

$a(4 - 2 + 1 - 3) + b(2 - 6 + 4) = 0$

$0 = 0$

For all values of $a$ and $b$ the plane

$a(2x - y + z - 3) + b(x - 3y + 4) = 0$

passes through the point $(2,2,1)$. This is because the given point lies on the line of intersection of the given planes.

*3. Since the second equation can be written

$3(2x - y + 3z) = 5$

it is clear that any triple in the solution set of the first equation (and therefore reducing the parenthesis to 1) will not be in the solution set of the second equation. Similarly for any triple in the solution set of the second equation. Thus the planes have no point in common and are parallel.

The equation

7 - 10e $a(2x - y + 3z - 1) + b(6x - 3y + 9z - 5) = 0$

can be written

$(a + 3b)(2x - y + 3z) + (-a - 5b) = 0$

If this plane passes through a point on the first plane, we know that

$2x - y + 3z = 1$

Therefore $a + 3b - a - 5b = 0$

$-2b = 0$

$b = 0$

Thus, any plane represented by (7 - 10e) that passes through a point in the first plane must coincide with the first plane. Similarly, if a plane represented by (7 - 10e) passes through a point of the second plane, we have

$$(a + 3b)\left(\frac{5}{3}\right) + (-a - 5b) = 0$$

$$2a = 0$$

$$a = 0$$

Therefore the plane coincides with the second plane.

We conclude that if $a \neq 0$ and $b \neq 0$, any plane represented by (7 - 10e) has no point in common with either of the given planes. It is therefore parallel to these planes.

*4. $a(x + y - 3) + b(z - 4) = 0$

Substituting $(1,-1,1)$ for $x,y,z$

$$a(-3) + b(-3) = 0; \quad a = -b.$$

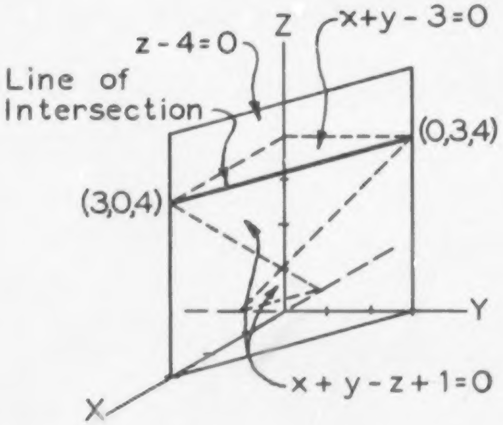
Take $a = 1$, $b = -1$

$$x + y - z + 1 = 0$$

The trace of this plane in the XZ plane is $x - z + 1 = 0$. This line intersects the XZ trace of

$$x + y - 3 = 0$$

which is $x - 3 = 0$. The point of intersection is $(3,0,4)$.



Similarly the trace of the plane, $x + y - z + 1 = 0$, in the YZ plane intersects the trace of $x + y - 3 = 0$ in the YZ plane in the point $(0,3,4)$. These points are both in the plane $Z = 4$. Thus the line joining these 2 points is the line of intersection of the three planes.

Exercises 7 - 11. Miscellaneous Exercises--Answers.

1. The number is 364.
2. $3x + 4y + 5z = a,$

   $4x + 5y + 6z = b,$

   $5x + 6y + 7z = c.$

   Eliminate x: $3x + 4y + 5z = a,$

   $y + z = 4a - 3b,$

   $2y + 2z = 5a - 3c.$

   Condition: $5a - 3c = 2(4a - 3b)$

   or $a + c = 2b$
3. The number is 456 or 654.
4. \$6500, \$1300, \$2200.
5. $a = 7$; The line is given by $\begin{cases} x = -2y + 7 \\ y \quad \text{arbitrary} \\ z = y - 1 \end{cases}$
6. 5 cu. yds., 6 cu. yds., 8 cu. yds.
7. 12 dimes, 8 nickels, 20 pennies.
8. 75 units, 80 units, 50 units.
9. 12 days, 8 days, 6 days.
10. 8 hours, 4 hours, 8 hours.

11. $AS = AR = 4\frac{1}{2}$; $BS = BT = 5\frac{1}{2}$; $CT = CR = 2\frac{1}{2}$

12. $y = 3x^2 + 2x - 1$

13. $y = -x^2 + 2x + 4$

14. 160 elementary school pupils

 80 high school pupils

 80 adults

15. $K = -8$; $A = 50$, $B = 0$

16. Rewrite the given equation

$w_1T + w_2Q + w_3E = (w_1 + w_2 + w_3)A$

as

$w_1(T - A) + w_2(Q - A) + w_3(E - A) = 0.$

Using the table of scores we construct the following table:

| | T - A | Q - A | E - A |
|---|---|---|---|
| Frank | -4 | -4 | 4 |
| Joyce | -2 | 18 | -6 |
| Eunice | 3 | -17 | 5 |

from which we write our system

$w_1 + w_2 - w_3 = 0,$

$w_1 - 9w_2 + 3w_3 = 0,$

$3w_1 - 17w_2 + 5w_3 = 0.$

Eliminate x:

$w_1 + w_2 - w_3 = 0,$

$-10w_2 + 4w_3 = 0,$

$-20w_2 + 8w_3 = 0.$

Each of the last two equations reduces to

$$5w_2 - 2w_3 = 0$$

So $w_2 = \frac{2}{5}w_3$ and $w_1 = w_3 - w_2 = \frac{3}{5}w_3$.

(Equivalently, $w_1 : w_2 : w_3 = 3:2:5$)

For $w_1 + w_2 + w_3 = 1$, we can write

$\frac{3}{5}w_3 + \frac{2}{5}w_3 + w_3 = 1$

or

$10w_3 = 5$ so $w_3 = 0.5$, $w_2 = 0.2$, $w_1 = 0.3$.

17. Let a = number of air mail stamps purchased
    f = number of 4 cent stamps purchased
    s = number of one cent stamps purchased

$.07a + .04f + .01s = 10$

$a = 2f$

observe that there are only 2 equations in 3 unknowns. However, s must be an integer less than 18 (the price of one 4 cent stamp and 2 air mail stamps) since only the change is spent for 1 cent stamps.

Substituting $a = 2f$ in

$7a + 4f + s = 1000$

we have $18f + s = 1000$

$s = 1000 - 18f$

$f = 55$ is the largest integral value that leaves s positive.

Therefore $f = 55$, $s = 10$, $a = 110$

Note that the problem can be solved simply by observing that we are to buy the largest number possible of 18 cent units (consisting of 2 air mail and 1 four cent stamp), and spend the change on 1 cent stamps.

18. Actual score is 81.

Reported score is 63.

Par is 60.

*19. If a, b, c, d are the coefficients of our desired plane,

$$ax + by + cz + d = 0,$$

we obtain three equations for the four "unknowns" a, b, c, d by demanding that the coordinates of the three given points shall satisfy this equation:

$$a\cdot(-1) + d = 0,$$
$$a.1 + b\cdot(-1) + d = 0,$$
$$a\cdot(-1) + b\cdot 3 + c\cdot 2 + d = 0;$$

or

$$-a + d = 0,$$
$$a - b + d = 0,$$
$$-a + 3b + 2c + d = 0.$$

Eliminating a from the second and third:

$$-a + d = 0,$$
$$-b + 2d = 0,$$
$$3b + 2c = 0.$$

Eliminating b from the third:

$$-a + d = 0,$$
$$-b + 2d = 0,$$
$$2c + 6d = 0.$$

Hence $a = d,\ b = 2d,\ c = -3d,$

an answer being $x + 2y - 3z + 1 = 0$ $(d = 1)$

Since any other choice of d will give an equation with coefficients proportional to these, only one plane is determined.